Green function and self-adjoint Laplacians on polyhedral surfaces

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Abstract

Green function and self-adjoint Laplacians on polyhedral surfaces

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Using Roelcke's formula for the Green function, we explicitly construct a basis in the kernel of the adjoint Laplacian on a compact polyhedral surface X and compute the S-matrix of X at the zero value of the spectral parameter. We apply these results to study various self-adjoint extensions of a symmetric Laplacian on a compact polyhedral surface of genus two with a single conical point. It turns out that the behaviour of the S-matrix at the zero value of the spectral parameter is sensitive to the geometry of the polyhedron.

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I would like to express my sincere gratitude to my supervisors, Alexey Kokotov and Victor Kalvin, for their guidance, support, commitment, trust, and words of wisdom during my Ph.D. studies at Concordia. Their knowledge and love for their profession have been my inspiration to continue and finish this research. This would not be possible without them.

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Lastly but not the least, I thank the Higher Being for giving me life.

To my wife, Eileen, and my baby, Kiel.

Contribution of the Authors

This thesis is a product of a research study whose main problem was conceptualized by Alexey Kokotov (AK), a professor at the Department of Mathematics and Statistics in Concordia University. It has been prepared in a "Manuscript-based" format. The main part of the thesis (Chapters 1 to 3) is based on the paper "Green function and self-adjoint Laplacians on polyhedral surfaces," which was first published in the *Canadian Journal of Mathematics* at https://doi.org/10.4153/S0008414X19000336.

Most results appearing in this thesis are joint work of Kelvin Lagota (KL) and AK. The proofs of following propositions were done by KL:

- *Asymptotics* (Proposition 2.1) - with the guidance of AK and Victor Kalvin;
- *Symplectic form* (Proposition 2.2);
- *expressions for the special growing solutions* (Proposition 2.8); and
- *comparison formulas* (Proposition 3.4) - joint work with AK.

These are found in Chapter 4 (the proof of the *Asymptotics* was outlined in the Appendix of the paper, the rest were not included).

Moreover, the "very symmetric case" (see Section 3.2.2) was initiated by KL. This has led to the conclusion of Proposition 3.12, whose proof was completed and furnished with the help of AK.

Table of Contents

List of Figures

List of Notations

X; X_0	compact Riemann surface, polyhedral surface; $X_0 := X \setminus \{P_1, \dots, P_M\}$
Δ	symmetric Laplace operator
$\bar{\Delta}$	closure of the Laplace operator with respect to the graph norm
Δ^*	adjoint of the Laplace operator
$\Delta_F/\Delta_{hol}/\Delta_{sing}$	Friedrichs/holomorphic/(maximal) singular (self-adjoint) extensions of Δ
$\ker A$	kernel of the operator A
$L^2(X)$	space of square Lebesgue integrable functions on X
$C_0^\infty(X_0)$	space of smooth functions with compact support in X
$\mathcal{D}(A)$	domain of the operator A
$\langle \cdot, \cdot \rangle$; $\| \cdot ; \mathcal{H} \|$	L^2-inner product; norm in $\mathcal{H}$
$G(x, y)$	Green function
$R(x, y; \lambda)$	resolvent kernel function
$\text{Area}(X)$	area of X
$\text{Re}(\lambda)$; $\text{Im}(\lambda)$	real part of λ; imaginary part of λ
$G_g(\cdot; \lambda)$	special growing solution with principal part g
$\det(A)$	determinant of the operator/matrix A
$\zeta_A(s) = \zeta(s, A)$	zeta-function of the operator A
$\dim(\mathcal{H})$	dimension of $\mathcal{H}$
$\widehat{u} = \mathcal{M}u$	Mellin transform of u
H^l	Sobolev space of order l
H^l_γ	weighted Sobolev space of order l with weight γ
$\mathscr{A}(\lambda)$	operator pencil
$K_\nu(\lambda)/I_\nu(\lambda)$	Bessel function of the first/third kind
$\text{spec}(A)$	spectrum of A

Chapter 1

Introduction

1.1 Background

The spectral geometry of a Riemannian manifold X with singularities is more involved than that of smooth manifolds, in particular, due to the following reason: it may happen that the symmetric Laplacian Δ (usually defined on smooth functions supported in $X \setminus \{singularities\}$) is not essentially self-adjoint, and, in order to consider the spectrum of the Laplacian, one has to make a choice from (infinitely) many possible self-adjoint extensions of Δ.

The case of Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ with punctures is investigated in great detail in [4] (see also the references therein), and manifolds of higher dimension with cone like singularities are also considered, for example, in the papers [16], [17], [25], [30] to mention a few. In this thesis, we consider the case of compact polyhedral surfaces (closed surfaces glued from Euclidean triangles). These are compact Riemann surfaces equipped with flat conformal metrics with conical singularities at the vertices of the corresponding polyhedron (it should be noted that the metric of a polyhedron does not see the edges: interior points of an edge are ordinary smooth point of the corresponding Riemannian manifold).

A question of general interest here can be formulated as follows: how do the spectral characteristics of the polyhedron depend on the choice of the self-adjoint extension of the symmetric Laplacian, the choice of conformal polyhedral metric, and the moduli of the underlying Riemann surface? This question was partially addressed in [13], where the dependence of an important spectral invariant, the ζ-regularized spectral determinant of the Laplacian, on the choice of the self-adjoint extension was analysed. It turned out that one can write a comparison formula for two determinants of the Laplacian corresponding to different self-adjoint extensions, and the main ingredient of this formula is the so-called S-matrix of the polyhedral surface. The S-matrix depends on a spectral parameter λ and is defined via the coefficients in the asymptotic expansions near the conical points of some special solutions (in classical sense) to the homogeneous Helmholtz equation $(\Delta - \lambda)u = 0$ on the polyhedron. Moreover, the behaviour of $S(\lambda)$ at the zero value of the spectral parameter plays especially important role; for instance, the order of the zero of a certain minor of $S(\lambda)$ at $\lambda = 0$ is related to the number of zero modes of the corresponding self-adjoint extension; most of the entries of the matrix $S(0)$ admit explicit expressions through holomorphic invariants of the underlying Riemann surface (Bergman kernel, Schiffer projective connection), and in case of a smooth surface with punctures (which can be considered as conical points of angle 2π), the entries of $S(0)$ are related to the Robin mass of the surface, etc.

1.2 Organization of the thesis

In this thesis, we apply and further develop the results of [13]. In Chapter 2, we discuss the general properties of the symmetric Laplacian Δ on an arbitrary polyhedral surface: we give an explicit description of the domain of its adjoint Δ^* and, in particular, explicitly construct a basis of the kernel $\ker \Delta^*$. Using the latter basis, we compute the matrix $S(0)$, expressing its entries via some holomorphic invariants of the underlying Riemann surface. Our main technical tool here is the Roelcke formula for the Green function of a closed surface which we briefly discuss in Section 2.1.

In Chapter 3, we apply the results of the previous chapter to the simplest example of a polyhedral surface, having (the lowest possible) genus two with one conical point. We study three concrete self-adjoint extensions of the symmetric Laplacian on this surface: the Friedrichs extension, the so-called holomorphic extension, and the maximal singular extension. Using the results of [13] and the explicit formulas for $S(0)$, we write down the precise (with all the auxiliary constants computed) comparison formulas relating the ζ-regularized determinants of these three extensions. It turns out that properties of the S-matrix depend on geometric properties of the polyhedral surface. We show that the dimension of the kernel of the holomorphic extension (related to the order of the zero of a certain minor of $S(\lambda)$) depends on the class of linear equivalence of the divisor $(2P)$, where P is the vertex of the polyhedron (this effect was previously found in [14], where the polyhedra of genus g with $2g-2$ vertices were considered), and that the dimension of the kernel of the maximal singular extension can be higher than usual if the surface has a very large group of symmetry.

Finally in Chapter 4, proofs of some results from Chapters 2 and 3 are provided.

Chapter 2

Green function and the kernel of the adjoint Laplacian for compact polyhedral surfaces

2.1 Roelcke's formula for the Green function

Let X be a compact Riemann surface endowed with a conformal metric $\mathbf{m}$; we assume that $\mathbf{m}$ is either smooth or flat with conical singularities. In the latter case, let $P_1, P_2, \dots, P_M$ be the conical singularities and denote $X_0 = X \setminus \{P_1, \dots, P_M\}$. Let Δ denote an unbounded densely-defined, symmetric operator in $L^2(X, \mathbf{m})$ with initial domain $C_0^\infty(X_0)$ and let $\overline{\Delta}$ be its closure whose domain $\mathcal{D}(\overline{\Delta})$ is the completion of $C_0^\infty(X_0)$ in the graph norm

$$\left(\left\|u; L^2(X, \mathbf{m})\right\|^2 + \left\|\Delta u; L^2(X, \mathbf{m})\right\|^2\right)^{1/2}. \tag{2.1}$$

We leave it to the readers to verify that this graph norm is equivalent to $\|u; L^2(X,\mathbf{m})\| + \|\Delta u; L^2(X,\mathbf{m})\|$. Let Δ^* be the adjoint of Δ in $L^2(X,\mathbf{m})$ (with initial domain $C_0^\infty(X_0)$) and denote by $\mathcal{D}(\Delta^*)$ its domain.

Let $\Delta^{\mathbf{m}}$ be the corresponding self-adjoint Laplace operator (in the case of conical metric, we define $\Delta^{\mathbf{m}}$ as the Friedrichs extension of the symmetric Laplace operator with domain consisting of smooth functions vanishing near the conical points: the functions from the domain of the Friedrichs extension are known to be bounded near the conical points), and let $G(x,y)$ be the Green function corresponding to $\Delta^{\mathbf{m}}$; this is defined to be the constant term of the Laurent expansion of the resolvent kernel function $R(x,y;\lambda)$ corresponding to $\Delta^{\mathbf{m}}$ at $\lambda = 0$:

$$R(x,y;\lambda) = -\frac{1}{\text{Area}(X)\lambda} + G(x,y) + O(\lambda). \tag{2.2}$$

The Green function is real-valued and satisfies the following properties:

1. $G(x,y) = G(y,x)$;
2. For $x \neq y$, $(\Delta^{\mathbf{m}})_x G(x,y) = (\Delta^{\mathbf{m}})_y G(x,y) = -\dfrac{1}{\text{Area}(X)}$;
3. $G(x,y) = -\dfrac{1}{2\pi}\log|x-y| + O(1)$ as $x \to y$;
4. In the case of conical metric, the Green function $G(\cdot, y)$ is bounded near all conical points (unless y itself is a conical point and the first argument approaches y);
5. For any $x \in X$, one has
$$\int_X G(x,y)\, dS(y) = 0, \tag{2.3}$$
where dS is the volume element of the metric $\mathbf{m}$.

In the case when the metric $\mathbf{m}$ is smooth, the Green function is given explicitly by the formula

$$G(x,y) = \frac{1}{2\pi\,\mathrm{Area}(X)^2}\int_X\int_X \mathrm{Re}\int_p^x \Omega_{y-q}\, dS(p)\, dS(q), \tag{2.4}$$

where Ω_{p-q} is the meromorphic one-form (2.5) below. The formula, which appeared in [8] (see equation (2.19) on page 31) is called there Roelcke's formula (without any reference). Unfortunately, we were unable to identify the primary source and it seems that [8] is the only published text containing this result in its full generality (it should be noted that the "Green function of a closed orientable surface" from [37], Section 4.2 is just the function F_{P_1,P_k} from Proposition 2.10 below and has nothing to do with the Green function discussed here). Formula (2.4) and its proof are also valid for conical metrics. For the reader's convenience, we decipher here the derivation of this formula given in passing in [8].

Choose a standard basis of a- and b-cycles on X. Let $\{v_j\}$ be the basis of the holomorphic one-forms on X that are normalized via $\int_{a_i} v_j = \delta_{ij}$. Let Ω_{p-q} be the meromorphic one-form (see, e.g., page 4 of [7] with a different normalization of basic holomorphic differentials) defined by

$$\Omega_{p-q}(z) = \int_p^q W(z,\cdot) - 2\pi i \sum_{\alpha,\beta} (\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta} v_\alpha(z)\, \mathrm{Im}\int_p^q v_\beta \tag{2.5}$$

where $\mathbb{B} = [\int_{b_i} v_j]$ is the matrix of b-periods and W is the canonical meromorphic bidifferential (see (3.17) below) on X. This one-form is the unique differential of the third kind with simple poles of residue -1 and 1 at p and q, respectively, and moreover, it has purely imaginary periods. Hence, the real part of the integral $\int_x^y \Omega_{p-q}$ does not depend on the path of integration and gives a harmonic function (with logarithmic singularities) with respect to the arguments x, y, p, q. Using the known singularities of the latter function, one can express it as

$$\operatorname{Re}\int_x^y \Omega_{p-q} = 2\pi\left(G(y,p) - G(y,q) + G(x,q) - G(x,p)\right). \tag{2.6}$$

Integrating (2.6) over X twice (first with respect to $dS(x)$ and then with respect to $dS(q)$), using (2.3), and renaming the arguments in the resulting expression, one obtains Roelcke's formula (2.4).

2.2 Harmonic functions with prescribed singularities

2.2.1 Domain of the self-adjoint operator

For a conical point P_j with conical angle β_j, let n_j be the integer such that $2\pi n_j < \beta_j \leq 2\pi(n_j + 1)$. In the proof of Proposition 2.7 below, a conical point with conical angle 2π will be considered. In this case, $n_j = 0$ and all the sums $\sum_{m=1}^{n_j}$ appearing in (2.7) in Proposition 2.1 are equal to 0 by definition. Introduce ζ_j to denote the *distinguished local parameter* near P_j: note that in the vicinity of P_j, one has

$$\mathbf{m}(\zeta_j, \bar{\zeta}_j)|d\zeta_j|^2 = |\zeta_j|^{2b_j}|d\zeta_j|^2$$

(see [13], Definition 1) and

$$\Delta^* = -4|\zeta_j|^{-2b_j}\partial_{\zeta_j}\partial_{\bar{\zeta}_j}$$

where $\beta_j = 2\pi(b_j+1)$. In polar coordinates (r,θ) where $r = \dfrac{|\zeta_j|^{b_j+1}}{b_j+1}$ and $\theta = \arg(\zeta_j)$, one can write

$$\mathbf{m}\,dx = dr^2 + (b_j+1)^2 r^2 d\theta^2 \quad \text{and} \quad \Delta^* = -\frac{1}{r^2}\left((r\partial_r)^2 + \frac{1}{(b_j+1)^2}\partial_\theta^2\right).$$

Proposition 2.1. *In the vicinity of the point P_j, a function $u \in \mathcal{D}(\Delta^*)$ has the asymptotics*

$$\begin{aligned} u \;&= \frac{i}{\sqrt{2\pi}}\mathfrak{L}_j(u)\log|\zeta_j| + \sum_{m=1}^{n_j}\frac{1}{\sqrt{4\pi m}}\mathfrak{H}_{j,m}(u)\frac{1}{\zeta_j^m} + \sum_{m=1}^{n_j}\frac{1}{\sqrt{4\pi m}}\mathfrak{A}_{j,m}(u)\frac{1}{\bar{\zeta}_j^m} \\ &+\frac{i}{\sqrt{2\pi}}\mathfrak{c}_j(u) + \sum_{m=1}^{n_j}\frac{1}{\sqrt{4\pi m}}\mathfrak{h}_{j,m}(u)\zeta_j^m + \sum_{m=1}^{n_j}\frac{1}{\sqrt{4\pi m}}\mathfrak{a}_{j,m}(u)\bar{\zeta}_j^m + \chi v, \end{aligned} \tag{2.7}$$

where χ is a smooth cut-off function that has compact support in a small vicinity of P_j and that is equal to 1 in a smaller vicinity, and v is a function from the domain of the closure $\mathcal{D}\left(\overline{\Delta}\right)$. One has the asymptotics $v = o(|\zeta_j|^{n_j})$ as $\zeta_j \to 0$.

The notation for the coefficients comes from the form of the corresponding term in the asymptotics: growing holomorphic ($\mathfrak{H}$), growing antiholomorphic ($\mathfrak{A}$), (growing) logarithmic ($\mathfrak{L}$), constant ($\mathfrak{c}$), decreasing holomorphic ($\mathfrak{h}$), and decreasing antiholomorphic ($\mathfrak{a}$). The normalizing factors $\left(\frac{1}{\sqrt{4\pi m}}, \frac{i}{\sqrt{2\pi}}, \text{etc.}\right)$ are introduced to obtain the standard Darboux basis for the symplectic form (2.9) below. The proof for the asymptotics (2.7) is given in Section 4.1.

2.2.2 Gelfand symplectic form

Let Ω be the symplectic form on the factor space $\mathcal{D}(\Delta^*)/\mathcal{D}(\overline{\Delta})$:

$$\Omega\left([u],[v]\right) := \langle \Delta^* u, \overline{v}\rangle - \langle u, \Delta^*\overline{v}\rangle\,, \tag{2.8}$$

where $\langle u,v\rangle=\int_X u\overline{v}\,dS$ is the usual hermitian product with volume element

$$dS=\mathbf{m}(\zeta,\bar\zeta)\frac{d\zeta\wedge d\bar\zeta}{-2i}=-\frac{1}{2i}\mathbf{m}(\zeta,\bar\zeta)|d\zeta|^2.$$

Straightforward computations (see Section 4.2) prove the next proposition:

Proposition 2.2. *One has*

$$\Omega\left([u],[v]\right)=\sum_{k=1}^{M}X_k(u)\begin{pmatrix}0 & -I_{2n_k+1}\\ I_{2n_k+1} & 0\end{pmatrix}X_k(v)^t \tag{2.9}$$

where $X_k(u)=(\mathfrak{L}_k(u),\mathfrak{H}_{k,1}(u),\dots,\mathfrak{H}_{k,n_k}(u),\mathfrak{A}_{k,1}(u),\dots,\mathfrak{A}_{k,n_k}(u),\mathfrak{c}_k(u),\mathfrak{h}_{k,1}(u),\dots,$ $\mathfrak{h}_{k,n_k}(u),\mathfrak{a}_{k,1}(u),\dots,\mathfrak{a}_{k,n_k}(u))$.

Remark 2.3. *Notice that an extension* Δ_E *is self-adjoint if* $\Omega\Big([u],\overline{[v]}\Big)=0$ *for all* $u,v\in\mathcal{D}(\Delta_E)$.

In fact, the Lagrangian (with respect to the form $(u,v)\mapsto\Omega([u],\overline{[v]})$) subspaces of the factor space $\mathcal{D}(\Delta^*)/\mathcal{D}(\bar\Delta)$ are in one-to-one correspondence with the self-adjoint extensions of $\bar\Delta$. Moreover, an extension can be defined by specifying conditions on the coefficients in the asymptotic expansion (2.7) of a given function $u\in\mathcal{D}(\Delta^*)$. For instance, the Friedrichs extension Δ_F is defined on functions $u\in\mathcal{D}(\Delta^*)$ not having the growing terms, i.e., $\mathfrak{L}_k(u)=\mathfrak{H}_{k,m}(u)=\mathfrak{A}_{k,m}(u)=0$ ([11], Proposition 3.5); or the *holomorphic extension* Δ_{hol} is defined on functions $u\in\mathcal{D}(\Delta^*)$ having only the holomorphic terms in their asymptotics, i.e., $\mathfrak{L}_k(u)=\mathfrak{A}_{k,m}(u)=\mathfrak{a}_{k,m}(u)=0$.

2.3 S-matrix of the polyhedral surface X

2.3.1 Special growing solutions

It is known that the kernel of the Friedrichs extension Δ_F has dimension 1 and consists of constants functions. For $\lambda \in \mathbb{C}$ not belonging to the spectrum of Δ_F, define for each $k = 1, \dots, M$ and $s = 1, \dots, n_k$ the unique *special growing solutions*

$$G_{1/\zeta_k^s}(\cdot;\lambda), \quad G_{1/\bar{\zeta}_k^s}(\cdot;\lambda), \quad G_{\log|\zeta_k|}(\cdot;\lambda) \tag{2.10}$$

of the homogeneous equation

$$\Delta^* u - \lambda u = 0 \tag{2.11}$$

via their asymptotic expansions near the conical points. More precisely, define G_{1/ζ_k^s} via

$$G_{1/\zeta_k^s}(\zeta_k;\lambda) = \frac{1}{\zeta_k^s} + O(1)$$

as $\zeta_k \to 0$ and

$$G_{1/\zeta_k^s}(x;\lambda) = O(1)$$

as $x \to P_l$ with $l \neq k$. Others are defined similarly.

Definition 2.4. *(See [13]) The constant terms and the coefficients of the powers of the decreasing terms ζ_k^s and $\bar{\zeta}_k^s$ ($k = 1, \dots, M$, $s = 1, \dots, n_k$) in the asymptotic expansions of the special growing solutions form the so-called S-matrix, $S(\lambda)$, of the surface X.*

For instance, the entry $S^{\frac{1}{\zeta_k^r}, \bar{\zeta}_l^s}(\lambda)$ of the S-matrix is given by the coefficient of the term $\bar{\zeta}_l^s$ in the asymptotic expansion of the special growing solution $G_{1/\zeta_k^r}(\cdot;\lambda)$ near the conical point P_l. Similarly, the entry $S^{\log|\zeta_k|, 1_l}(\lambda)$ is the constant term in the asymptotic expansion of the special growing solution $G_{\log|\zeta_k|}(\cdot;\lambda)$ near the conical point P_l.

The next proposition is a slightly improved version of Proposition 7 in [12]:

Proposition 2.5. *All the entries of the matrix $S(\lambda)$ except $S^{\log|\zeta_k|,1_l}(\lambda)$ admit holomorphic continuation to $\lambda = 0$; the entries $S^{\log|\zeta_k|,1_l}(\lambda)$ have a simple pole at $\lambda = 0$.*

Proof. We start with reminding the reader the construction of the special growing solutions (2.10). Let F be one of the following functions defined on the whole X:

$$\chi \log|\zeta_k|, \quad \chi\frac{1}{\zeta_k^l}, \quad \chi\frac{1}{\bar{\zeta}_k^l},$$

where χ is a smooth cut-off function supported in a small vicinity of P_k such that $\chi = 1$ in some smaller vicinity of P_k. Let λ do not belong to the spectrum of Δ_F. Introduce the function

$$f := (\Delta^* - \lambda)F$$

and define $g(\cdot;\lambda)$ as the (unique) solution of the equation

$$(\Delta_F - \lambda)g = (\Delta^* - \lambda)F \tag{2.12}$$

(it should be noticed that the right-hand side of this equation belongs to $L^2(X, \mathbf{m})$). Then

$$G(\cdot;\lambda) = F(\cdot) - g(\cdot;\lambda)$$

is the special growing solution with principal part F. It follows from the above construction that

$$
\begin{aligned}
g(\cdot;\lambda) &= g(\cdot;\lambda) + \frac{1}{\text{Area}(X)\lambda}\int_X f(\cdot;\lambda) - \frac{1}{\text{Area}(X)\lambda}\int_X f(\cdot;\lambda) \\
&= \left[(\Delta_F - \lambda)\Big|_{1^\perp}\right]^{-1}\left((\Delta^* - \lambda)F - \frac{1}{\text{Area}(X)}\int_X (\Delta^* - \lambda)F\right) \qquad (2.13) \\
&\quad - \frac{1}{\text{Area}(X)\lambda}\int_X f(\cdot;\lambda).
\end{aligned}
$$

The first term in (2.13) is holomorphic in a vicinity of the point $\lambda = 0$ (a simple eigenvalue of Δ_F). The behaviour of the second term at $\lambda = 0$ depends on the choice of the principal part F. In the case of $F = \chi\frac{1}{\zeta_k^l}$ or $\chi\frac{1}{\bar{\zeta}_k^l}$, the second term is again holomorphic at $\lambda = 0$, thanks to the obvious relation

$$
\int_X f(\cdot;0) = 0\,. \qquad (2.14)
$$

If the principal part F is logarithmic ($F = \chi\log|\zeta_k|$), then (2.14) is no longer true and $g(\cdot;\lambda)$ has a simple pole with residue

$$
-\frac{1}{\text{Area}(X)}\int_X \Delta^* F = -\frac{2\pi}{\text{Area}(X)}
$$

Summing up, the special growing solution $G_{1/\zeta_k^l}(\cdot;\lambda)$ and $G_{1/\bar{\zeta}_k^l}(\cdot;\lambda)$ are holomorphic with respect to λ at $\lambda = 0$, whereas

$$
G_{\log|\zeta_k|}(\cdot;\lambda) = \frac{2\pi}{\text{Area}(X)\lambda} + h(\cdot;\lambda)
$$

where $h(\cdot;\lambda)$ is holomorphic near $\lambda = 0$. Thus, all the coefficients in the asymptotic expansion of $G_{1/\zeta_k^l}(\cdot;\lambda)$ and $G_{1/\bar{\zeta}_k^l}(\cdot;\lambda)$ are holomorphic at $\lambda = 0$; the constant term in the asymptotics $G_{\log|\zeta|}(\cdot;\lambda)$ blows up at $\lambda = 0$, all other coefficients in the asymptotics $G_{\log|\zeta|}(\cdot;\lambda)$ are holomorphic at λ=0. □

Remark 2.6. *The values at $\lambda = 0$ of nonsingular entries of the S-matrix do depend on the choice of a metric* $\mathbf{m}$ *within a given conformal class through their dependence on the distinguished local parameters of the metric near the conical points. The opposite statement in Proposition 7 from [12] was made under an implicit assumption that the conformal factor is equal to one in small vicinities of the conical points.*

The values of the nonsingular entries of the S-matrix at $\lambda = 0$ can be found from the asymptotics of the (unique) special growing solutions $G_{1/\zeta_k^l}(\cdot;0)$, $G_{1/\bar{\zeta}_k^l}(\cdot;0)$ of the equation

$$\Delta^* u = 0 \tag{2.15}$$

subject to the condition

$$\int_X u dS = 0\,. \tag{2.16}$$

It should be noted that there is no harmonic function on X with a single logarithmic singularity, so the special growing solutions $G_{\log|\zeta_k|}(\cdot;0)$ *do not exist.* The following proposition gives the first new results. A closely related statement for the Green functions of elliptic boundary value problems in domains with conical points at the boundaries can be found in [28].

Proposition 2.7. *Let $y \in X \setminus \{P_1, \dots, P_M\}$.*

1. *The special growing solutions $G_{1/\zeta_k^l}(y;0)$, $G_{1/\bar{\zeta}_k^l}(y;0)$, $l = 1, \dots, n_k$, of the equation (2.15) are related to the coefficients of the asymptotic expansion of the Green function $G(\cdot, y)$ at the conical point P_k via*

$$G(\zeta_k, y) = G(P_k, y) - \sum_{l=1}^{n_k} \frac{1}{4\pi l} G_{1/\zeta_k^l}(y;0)\zeta_k^l - \sum_{l=1}^{n_k} \frac{1}{4\pi l} G_{1/\bar{\zeta}_k^l}(y;0)\bar{\zeta}_k^l + o(|\zeta_k|^{n_k})\,. \tag{2.17}$$

2. The constant term, $G(P_k, y)$, in (2.17) can be represented as

$$G(P_k, y) = \frac{1}{2\pi} \lim_{\lambda \to 0} \left[G_{\log|\zeta_k|}(y; \lambda) - \frac{2\pi}{\text{Area}(X)\lambda} \right] . \tag{2.18}$$

Proof. Until the end of this proof, assume that $X_0 = X \setminus \{P_1, \dots, P_M, y\}$, that is, the point y is considered a conical point with conical angle 2π. Then $G(\cdot, y)$ belongs to the domain of the operator Δ^*; the latter operator is now the adjoint to the symmetric Laplacian with domain $C_0^\infty(X \setminus \{P_1, \dots, P_M, y\})$.

It should be noticed that the functions u from $\mathcal{D}(\Delta^*)$ have the asymptotics

$$u(\zeta(x)) = \frac{i}{\sqrt{2\pi}} \mathfrak{L}_y(u) \log|\zeta| + \frac{i}{\sqrt{2\pi}} \mathfrak{c}_y(u) + o(1)$$

as $x \to y$ (here the local parameter ζ is defined via $\mathbf{m} = |d\zeta|^2$ near y and $\zeta(y) = 0$). Since $\int_X G_{1/\zeta_k^l} dS = 0$ and $\Delta_x^* G(x, y) = -\frac{1}{\text{Area}(X)}$ (a constant), one has

$$\Omega([G(\cdot, y)], [G_{1/\zeta_k^l}(y; 0)]) = 0 . \tag{2.19}$$

On the other hand, (2.9) implies

$$\begin{aligned} \Omega([G(\cdot, y)], [G_{1/\zeta_k^l}(y; 0)]) &= \mathfrak{h}_{k,l}(G(\cdot, y)) \mathfrak{H}_{k,l}(G_{1/\zeta_k^l}(\cdot; 0)) - \mathfrak{L}_y(G(\cdot, y)) \mathfrak{c}_y(G_{1/\zeta_k^l}(\cdot; 0)) \\ &= \sqrt{4\pi l}\, \mathfrak{h}_{k,l}(G(\cdot, y)) - \frac{1}{\sqrt{2\pi}\, i} \left[\frac{\sqrt{2\pi}}{i} G_{1/\zeta_k^l}(y; 0) \right] \end{aligned}$$

and, therefore,

$$\mathfrak{h}_{k,l}(G(\cdot, y)) = -\frac{1}{\sqrt{4\pi l}} G_{1/\zeta_k^l}(y; 0) .$$

Similarly,

$$\mathfrak{a}_{k,l}(G(\cdot, y)) = -\frac{1}{\sqrt{4\pi l}} G_{1/\bar{\zeta}_k^l}(y; 0),$$

and (2.17) follows.

To show (2.18), let $R(x,y;\lambda)$ be the resolvent kernel of Δ_F. Consider the expression

$$E(\lambda) = \left\langle (\Delta^* - \lambda)\left[R(\cdot,y;\lambda) + \frac{1}{\mathrm{Area}(X)\lambda}\right],\ G_{\log|\zeta_k|}(\cdot;\lambda) - \frac{2\pi}{\mathrm{Area}(X)\lambda}\right\rangle - \left\langle R(\cdot,y;\lambda) + \frac{1}{\mathrm{Area}(X)\lambda},\ (\Delta^* - \lambda)\left[G_{\log|\zeta_k|}(\cdot;\lambda) - \frac{2\pi}{\mathrm{Area}(X)\lambda}\right]\right\rangle \tag{2.20}$$

Since $\lim_{\lambda\to 0}\left[R(\cdot,y;\lambda) + \frac{1}{\mathrm{Area}(X)\lambda}\right] = G(\cdot,y)\perp 1$ and

$$\int_X \left[G_{\log|\zeta_k|}(\cdot;\lambda) - \frac{2\pi}{\mathrm{Area}(X)\lambda}\right] = 0,\ ^{1}$$

one has $E(\lambda) = o(1)$ as $\lambda \to 0$. On the other hand, computing $E(\lambda)$ via (2.9), one gets

$$E(\lambda) = \left[G_{\log|\zeta_k|}(y;\lambda) - \frac{2\pi}{\mathrm{Area}(X)\lambda}\right] - 2\pi\left[R(P_k,y;\lambda) + \frac{1}{\mathrm{Area}(X)\lambda}\right]$$

which implies (2.18). □

The next proposition immediately follows from (2.17), (2.18), and Roelcke's formula (2.4). See Section 4.3 for the details.

Proposition 2.8. *One has the following explicit expressions for the special growing solutions of the homogeneous Laplace equation* (2.15) *subject to* (2.16)*:*

$$G_{1/\zeta_k^l}(y;0) = -\frac{1}{(l-1)!\,\mathrm{Area}(X)}\int_X \Omega_{y-q}^{(l-1)}(P_k)\,dS(q) \qquad l = 1,\dots,n_k, \tag{2.21}$$

$$G_{1/\bar\zeta_k^l}(y;0) = \overline{G_{1/\zeta_k^l}(y;0)}\,. \tag{2.22}$$

Moreover, one has the relation

$$\lim_{\lambda\to 0}\left[G_{\log|\zeta_k|}(y;\lambda) - \frac{2\pi}{\mathrm{Area}(X)\lambda}\right] = \frac{1}{\mathrm{Area}(X)^2}\int_X\int_X \mathrm{Re}\int_p^{P_k} \Omega_{y-q}\ dS(p)dS(q)\,. \tag{2.23}$$

[1] The equality can be checked as follows:

$$\int_X G = \int_X (F-g) = \int_X F - \left(\frac{1}{\lambda}\int_X \Delta^* g - \frac{1}{\lambda}\int_X f\right) = \int_X F + \frac{1}{\lambda}\int_X f = \frac{1}{\lambda}\int_X \Delta^* F = \frac{2\pi}{\lambda}$$

In (2.21), the expression $\Omega_{y-q}^{(l-1)}(P_k)$ should be understood as follows. Write the one form Ω_{y-q} in the distinguished local parameter ζ_k in a vicinity of the conical point P_k:

$$\Omega_{y-q} = \omega(\zeta_k)d\zeta_k\,.$$

Then

$$\Omega_{y-q}^{(l-1)}(P_k) := \left(\frac{d}{d\zeta_k}\right)^{l-1} \omega(\zeta_k)|_{\zeta_k=0}\,.$$

2.3.2 Explicit expressions for $S(0)$

Rewriting Ω_{y-q} as

$$\Omega_{y-q}(z) = \int_y^q W(z,\,\cdot\,) - \pi \sum_{\alpha,\beta=1}^{g} (\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta}\, v_\alpha(z) \int_y^q v_\beta + \pi \sum_{\alpha,\beta=1}^{g} (\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta}\, v_\alpha(z) \overline{\int_y^q v_\beta} \qquad (2.24)$$

and using in (2.23) the reciprocity law for normalized differentials of the third kind

$$\mathrm{Re} \int_S^R \Omega_{P-Q} = \mathrm{Re} \int_Q^P \Omega_{R-S}$$

(see, e.g., [6], p. 67), one can easily find all the terms of the asymptotic expansions of $G_{1/\zeta_k^l}(y;0)$ and $\lim_{\lambda\to 0}\left[G_{\log|\zeta_k|}(y;\lambda) - \frac{2\pi}{\mathrm{Area}(X)\lambda}\right]$ as $y \to P_l$, $l = 1, \dots, M$. This results in explicit formulas for all the finite entries of the matrix $S(0)$. For instance, (2.23) and the reciprocity law immediately imply that

$$S^{\log|\zeta_k|,\ \zeta_l}(0) = \frac{1}{2\,\mathrm{Area}(X)} \int_X \Omega_{P_k-p}(P_l) dS(p),\ \ l \neq k. \qquad (2.25)$$

Similarly, from (2.21) and (2.24), one gets the relation

$$S^{\frac{1}{\zeta_k},\ \bar\zeta_l}(0) = \pi \sum_{\alpha,\beta=1}^{g} (\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta} v_\alpha(P_k)\overline{v_\beta(P_l)} = \pi B(P_k, P_l)\,, \qquad (2.26)$$

where B is the Bergman reproducing kernel for holomorphic differentials (see, e.g., [8], equation (1.25)). (Here the value of a differential at P_l means its value in the distinguished local parameter at this point.) Following [7] and [42], introduce the Schiffer bidifferential on X as

$$\mathcal{S}(P,Q) = W(P,Q) - \pi \sum_{\alpha,\beta=1}^{g} (\operatorname{Im}\mathbb{B})^{-1}_{\alpha\beta}\, v_\alpha(P) v_\beta(Q)\,. \tag{2.27}$$

The Schiffer projective connection, S_{Sch}, is defined via the asymptotics of the Schiffer bidifferential at the diagonal $P = Q$:

$$\frac{\mathcal{S}(x(P),x(Q))}{dx(P)dx(Q)} = \frac{1}{(x(P)-x(Q))^2} + \frac{1}{6} S_{Sch}(x(P)) + O(x(P)-x(Q))\,, \tag{2.28}$$

as $Q \to P$. From (2.21) and (2.24) together with (2.27) and (2.28), one gets (cf. [14])

$$S^{\frac{1}{\zeta_k},\ \zeta_l}(0) = -\mathcal{S}(P_k, P_l); \quad l \neq k, \tag{2.29}$$

and

$$S^{\frac{1}{\zeta_k},\ \zeta_k}(0) = -\frac{1}{6} S_{Sch}(\zeta_k)\Big|_{\zeta_k=0}\,. \tag{2.30}$$

In the same manner, one can find explicit expressions for all the remaining (finite) entries of $S(0)$.

Remark 2.9. *It looks natural to define the regularized values of the singular entries of* $S(\lambda)$ *at* $\lambda = 0$ *via*

$$\operatorname{reg} S^{\log|\zeta_k|,\ 1_l}(0) := \lim_{\lambda\to 0} \left(S^{\log|\zeta_k|,\ 1_l}(\lambda) - \frac{2\pi}{\operatorname{Area}(X)\lambda} \right)\,. \tag{2.31}$$

In the case of a smooth surface X *with a puncture* P*, considered as a conical point of angle* 2π *(see, e.g., [43], [2]), the special growing solution* $G_{\log d(\cdot,P)}(\cdot;\lambda)$ *coincides with* $2\pi\, R(\cdot,P;\lambda)$*, where* R *is the resolvent kernel of the Friedrichs extension of the Laplacian* *on $X \setminus \{P\}$ and d is the geodesic distance on X; the above regularization of a (single) entry of $S(0)$ coincides with $2\pi\, m(P)$, where $m(P)$ is the so-called Robin's mass (see, e.g., [39], [32])*

$$\operatorname{reg} S^{\log d(P,\cdot),1}(0) = m(P) = \lim_{Q\to P}\Big(G(P,Q) + \frac{1}{2\pi}\log d(P,Q)\Big)\,.$$

In particular, formula (2.23) leads to an explicit expression for $m(P)$. Unfortunately, the latter expression contains the finite part of a diverging line integral and, therefore, is not as that effective as formulas (2.25), (2.26), (2.29), and (2.30). It should be noticed that using the technique of string theorists ([33], [44]), one can get a nice expression for the centered Robin's mass

$$m(P) - \frac{M(X)}{\operatorname{Area}(X)},$$

where

$$M(X) = \int_X m(P)dS(P).$$

Following [44], define the function Φ on $X\times X$ via

$$-4\pi\Phi(z,w) := -2\pi\left[\int_w^z \overrightarrow{v}\right]^t (\operatorname{Im}\mathbb{B})^{-1}\operatorname{Im}\int_w^z \overrightarrow{v} + \log\left(|E(z,w)|^2(\rho(z)\rho(w))^{1/2}\right)\,.$$

Here $\rho(z,\bar z)|dz|^2$ is the (smooth) metric on X and $E(z,w)$ is the prime form (see, e.g., [7]), $\overrightarrow{v} = (v_1,\dots,v_g)^t$. The results from Section 5 of [44] imply the relation

$$-G(z,w) + \frac{1}{2}m(z) + \frac{1}{2}m(w) = \Phi(z,w) + C \tag{2.32}$$

with some constant C. Integrating (2.32), one gets

$$\frac{M(X)}{2} + \frac{1}{2}m(w)\operatorname{Area}(X) = \int_X \Phi(z,w)dS(z) + C\operatorname{Area}(X) \tag{2.33}$$

and, therefore,

$$M(X)\,\mathrm{Area}(X)=\iint_{X\times X}\Phi(z,w)dS(z)dS(w)+C\,\mathrm{Area}(X)^2\,.$$

This gives the following explicit expression for centered Robin's mass:

$$\begin{aligned} m(w)-\frac{M(X)}{\mathrm{Area}(X)}=&\frac{2}{\mathrm{Area}(X)}\int_X\Phi(z,w)dS(z)\\ &-\frac{2}{\mathrm{Area}(X)^2}\iint_{X\times X}\Phi(z,w)dS(z)dS(w)\,. \end{aligned} \tag{2.34}$$

Moreover, from (2.32) and (2.33) follows an interesting counterpart of Roelcke formula (2.4)

$$G(z,w)=\frac{1}{2}\left(m(z)-\frac{M(X)}{\mathrm{Area}(X)}\right)+\frac{1}{\mathrm{Area}(X)}\int_X\Phi(z,w)dS(z)-\Phi(z,w), \tag{2.35}$$

mentioned in the last lines of Section 5 of [44].

2.3.3 Kernel of Δ^*

Motivated by the recent paper [26], we shall write down the basis in the kernel of the adjoint operator Δ^* (we remind the reader that Δ is the symmetric Laplacian with domain $C_0^\infty(X_0)$). This makes the constructions from Theorem 1 in [26] more explicit.

Putting $v=1$ in (2.9), one gets

$$\sum_{k=1}^{M}\mathfrak{L}_k(u)=0 \tag{2.36}$$

for any $u\in\ker(\Delta^*)$. On the other hand, for any two points P and Q of X, there exists a harmonic function u on $X\setminus\{P,Q\}$ with asymptotics $u(x)=\log d(x,P)+O(1)$ as $x\to P$ and $u(x)=-\log d(x,Q)+O(1)$ as $x\to Q$. Thus, Proposition 2.8 and the equality $\ker(\Delta_F) = \{\mathrm{const}\}$ imply the following statement:

Proposition 2.10. *The basis of* $\ker(\Delta^*)$ *consists of*

1. 1;

2. *functions* $G_{1/\zeta_k^l}(\cdot;0)$; $k = 1, \dots, M; l = 1, \dots, n_k$ *from Proposition 2.8;*

3. *functions* $G_{1/\bar{\zeta}_k^l}(\cdot;0)$; $k = 1, \dots, M; l = 1, \dots, n_k$ *from Proposition 2.8; and*

4. *functions* $F_{P_1,P_k}(P) = \mathrm{Re} \int^P \Omega_{P_1-P_k}$; $k = 2, \dots, M$, *where* $\Omega_{P_1-P_k}$ *is the meromorphic one form from (2.5).*

Chapter 3

Self-adjoint Laplacians on genus two polyhedral surfaces with one conical point

3.1 Comparison formulas for $\det\Delta_{hol}$ and $\det\Delta_{sing}$

In this section, several applications of the results of the previous chapter will be considered, particularly, to concrete classes of polyhedral surfaces. In order to avoid unnecessary technical complications, the simplest case of genus two surfaces with a single conical point P of conical angle 6π is studied. Thus, using the setting of Section 2.2.1, one has $M = 1$, $n_1 = 2$, $\beta := \beta_1 = 6\pi$,

$$\Omega([u],[v]) = X(u) \begin{pmatrix} 0 & -I_5 \\ I_5 & 0 \end{pmatrix} X(v)^t, \tag{3.1}$$

$$X(u) = (\mathfrak{L}(u), \mathfrak{H}_1(u), \mathfrak{H}_2(u), \mathfrak{A}_1(u), \mathfrak{A}_2(u), \mathfrak{c}(u), \mathfrak{h}_1(u), \mathfrak{h}_2(u), \mathfrak{a}_1(u), \mathfrak{a}_2(u)),$$

and the asymptotics in the vicinity of the point P of a function u from $\mathcal{D}(\Delta^*)$ in the distinguished local parameter ζ has the form

$$u = \frac{1}{\sqrt{8\pi}}\mathfrak{H}_2(u)\frac{1}{\zeta^2} + \frac{1}{\sqrt{8\pi}}\mathfrak{A}_2(u)\frac{1}{\bar{\zeta}^2} + \frac{1}{\sqrt{4\pi}}\mathfrak{H}_1(u)\frac{1}{\zeta} + \frac{1}{\sqrt{4\pi}}\mathfrak{A}_1(u)\frac{1}{\bar{\zeta}} + \frac{i}{\sqrt{2\pi}}\mathfrak{L}(u)\log|\zeta|$$

$$+ \frac{i}{\sqrt{2\pi}}\mathfrak{c}(u) + \frac{1}{\sqrt{4\pi}}\mathfrak{h}_1(u)\zeta + \frac{1}{\sqrt{4\pi}}\mathfrak{a}_1(u)\bar{\zeta} + \frac{1}{\sqrt{8\pi}}\mathfrak{h}_2(u)\zeta^2 + \frac{1}{\sqrt{8\pi}}\mathfrak{a}_2(u)\bar{\zeta}^2 + v$$

with $v = o(|\zeta|^2)$.

The following three regular[1] self-adjoint extensions of the symmetric Laplacian Δ with domain $C_0^\infty(X \setminus \{P\})$ will be considered:

- the Friedrichs extension Δ_F corresponding to the Lagrangian subspace of $\mathcal{D}(\Delta^*)/\mathcal{D}(\bar{\Delta})$

$$\mathfrak{L}(\cdot) = \mathfrak{H}_1(\cdot) = \mathfrak{H}_2(\cdot) = \mathfrak{A}_1(\cdot) = \mathfrak{A}_2(\cdot) = 0\,,$$

- the maximal singular extension Δ_{sing} corresponding to the Lagrangian subspace

$$\mathfrak{L}(\cdot) = \mathfrak{h}_1(\cdot) = \mathfrak{h}_2(\cdot) = \mathfrak{a}_1(\cdot) = \mathfrak{a}_2(\cdot) = 0\,,$$

- the holomorphic extension Δ_{hol} corresponding to the Lagrangian subspace

$$\mathfrak{L}(\cdot) = \mathfrak{A}_1(\cdot) = \mathfrak{A}_2(\cdot) = \mathfrak{a}_1(\cdot) = \mathfrak{a}_2(\cdot) = 0\,.$$

Proposition 3.1. *The operators $(\Delta_{sing}-\lambda)^{-1}-(\Delta_F-\lambda)^{-1}$ and $(\Delta_{hol}-\lambda)^{-1}-(\Delta_F-\lambda)^{-1}$ are finite dimensional, and one has the following representations for their traces:*

[1] A **regular extension** is one that is defined on a class of functions $u \in \mathcal{D}(\Delta^*)$ such that u does not have any logarithmic term in its asymptotic expansion ([13], Definition 5.2).

$$\mathrm{Trace}[(\Delta_{sing}-\lambda)^{-1}-(\Delta_F-\lambda)^{-1}] = -\,\mathrm{Trace}\left(T^{-1}(\lambda)T'(\lambda)\right) \tag{3.2}$$

$$\mathrm{Trace}[(\Delta_{hol}-\lambda)^{-1}-(\Delta_F-\lambda)^{-1}] = -\,\mathrm{Trace}\left(P^{-1}(\lambda)P'(\lambda)\right)\ , \tag{3.3}$$

where the matrices $T(\lambda)$ *and* $P(\lambda)$ *are given in* (3.6) *and* (3.16) *below.*

Proof. Notice that the kernel of the operator $\Delta^*-\lambda$ with $\lambda\in\mathbb{C}\setminus\mathrm{spec}(\Delta_F)$ is generated by the special growing solutions

$$G_{1/\zeta^2}(\cdot;\lambda),\ G_{1/\bar\zeta^2}(\cdot;\lambda),\ G_{1/\zeta}(\cdot;\lambda),\ G_{1/\bar\zeta}(\cdot;\lambda),\ G_{\log|\zeta|}(\cdot;\lambda)$$

of the equation $\Delta^*u-\lambda u=0$ and, therefore, the deficiency indices of Δ are $(5,5)$. So, the Krein formula for the difference of the resolvents of two self-adjoint extensions of a symmetric operator with (equal) finite deficiency indices can be applied (see Appendix A for a brief discussion; see also, e.g., [3], Vol. 2, Section 84): given $f\in L^2(X)$,

$$\begin{aligned}[(\Delta_{sing}-\lambda)^{-1}-(\Delta_F-\lambda)^{-1}](f) &= \sum_\alpha G_\alpha(\cdot;\lambda)\sum_\beta x_{\alpha\beta}(\lambda)\left\langle f, G_\beta(\cdot,\bar\lambda)\right\rangle \\ &= \sum_\alpha G_\alpha(\cdot;\lambda)X_\alpha(\lambda)\end{aligned} \tag{3.4}$$

where $\alpha=1/\zeta^2,\ 1/\zeta,\ 1/\bar\zeta^2,\ 1/\bar\zeta$, and $\beta=1/\zeta^2,\ 1/\zeta,\ 1/\bar\zeta^2,\ 1/\bar\zeta,\ \log|\zeta|$. Introducing $u\in\mathcal{D}(\Delta_F)$ via $(\Delta_F-\lambda)u=f$ and comparing the coefficients in the asymptotic expansion of the left- and right-hand sides of (3.4), one gets

$$-\Big(\frac{1}{\sqrt{4\pi}}\mathfrak{h}_1(u), \frac{1}{\sqrt{8\pi}}\mathfrak{h}_2(u), \frac{1}{\sqrt{4\pi}}\mathfrak{a}_1(u), \frac{1}{\sqrt{8\pi}}\mathfrak{a}_2(u)\Big)^t = T(\lambda)\left(X_{1/\zeta}, X_{1/\zeta^2}, X_{1/\bar\zeta}, X_{1/\bar\zeta^2}\right)^t \tag{3.5}$$

with

$$T(\lambda) = \begin{pmatrix} S^{1/\zeta,\ \zeta}(\lambda) & S^{1/\zeta^2,\ \zeta}(\lambda) & S^{1/\bar\zeta,\ \zeta}(\lambda) & S^{1/\bar\zeta^2,\ \zeta}(\lambda) \\ S^{1/\zeta,\ \zeta^2}(\lambda) & S^{1/\zeta^2,\ \zeta^2}(\lambda) & S^{1/\bar\zeta,\ \zeta^2}(\lambda) & S^{1/\bar\zeta^2,\ \zeta^2}(\lambda) \\ S^{1/\zeta,\ \bar\zeta}(\lambda) & S^{1/\zeta^2,\ \bar\zeta}(\lambda) & S^{1/\bar\zeta,\ \bar\zeta}(\lambda) & S^{1/\bar\zeta^2,\ \bar\zeta}(\lambda) \\ S^{1/\zeta,\ \bar\zeta^2}(\lambda) & S^{1/\zeta^2,\ \bar\zeta^2}(\lambda) & S^{1/\bar\zeta,\ \bar\zeta^2}(\lambda) & S^{1/\bar\zeta^2,\ \bar\zeta^2}(\lambda) \end{pmatrix}. \tag{3.6}$$

Since (3.5) holds with an arbitrary left-hand side (one can take as u an arbitrary function from $\mathcal{D}(\Delta_F)$), the matrix $T(\lambda)$ is invertible.

Notice that

$$\begin{aligned} \left\langle(\Delta^* - \lambda)u, G_{1/\bar\zeta}(\cdot;\bar\lambda)\right\rangle &= \left\langle(\Delta^* - \lambda)u, \overline{G_{1/\zeta}(\cdot;\lambda)}\right\rangle - \left\langle u, \overline{(\Delta^* - \lambda)G_{1/\zeta}(\cdot;\lambda)}\right\rangle \\ &= \Omega(u, G_{1/\zeta}(\cdot;\lambda)) = \sqrt{4\pi}\mathfrak{h}_1(u)\,. \end{aligned} \tag{3.7}$$

Similarly,

$$\left\langle(\Delta^* - \lambda)u, G_{1/\zeta}(\cdot;\bar\lambda)\right\rangle = \sqrt{4\pi}\mathfrak{a}_1(u), \tag{3.8}$$

$$\left\langle(\Delta^* - \lambda)u, G_{1/\bar\zeta^2}(\cdot;\bar\lambda)\right\rangle = \sqrt{8\pi}\mathfrak{h}_2(u), \tag{3.9}$$

and

$$\left\langle(\Delta^* - \lambda)u, G_{1/\zeta^2}(\cdot;\bar\lambda)\right\rangle = \sqrt{8\pi}\mathfrak{a}_2(u)\,. \tag{3.10}$$

Meanwhile, differentiating (2.12) with respect to λ and using (3.7)-(3.10), one gets

$$\begin{aligned} \frac{d}{d\lambda}S^{\star,\zeta}(\lambda) &= \frac{1}{4\pi}\left\langle G_\star(\cdot,\lambda), G_{1/\bar\zeta}(\cdot;\bar\lambda)\right\rangle, \\ \frac{d}{d\lambda}S^{\star,\zeta^2}(\lambda) &= \frac{1}{8\pi}\left\langle G_\star(\cdot,\lambda), G_{1/\bar\zeta^2}(\cdot;\bar\lambda)\right\rangle, \\ \frac{d}{d\lambda}S^{\star,\bar\zeta}(\lambda) &= \frac{1}{4\pi}\left\langle G_\star(\cdot,\lambda), G_{1/\zeta}(\cdot;\bar\lambda)\right\rangle, \\ \frac{d}{d\lambda}S^{\star,\bar\zeta^2}(\lambda) &= \frac{1}{8\pi}\left\langle G_\star(\cdot,\lambda), G_{1/\zeta^2}(\cdot;\bar\lambda)\right\rangle, \end{aligned} \tag{3.11}$$

with $\star = 1/\zeta, 1/\bar{\zeta}, 1/\zeta^2, 1/\bar{\zeta}^2$. Now, (3.4) can be rewritten as

$$\Big[(\Delta_{sing}-\lambda)^{-1}-(\Delta_F-\lambda)^{-1}\Big](f)$$
$$= -\Big(G_{1/\zeta}(\cdot;\lambda), G_{1/\zeta^2}(\cdot;\lambda), G_{1/\bar{\zeta}}(\cdot;\lambda), G_{1/\bar{\zeta}^2}(\cdot;\lambda)\Big)\, T^{-1}(\lambda)\times$$
$$\Big(\frac{1}{\sqrt{4\pi}}\mathfrak{h}_1(u), \frac{1}{\sqrt{8\pi}}\mathfrak{h}_2(u), \frac{1}{\sqrt{4\pi}}\mathfrak{a}_1(u), \frac{1}{\sqrt{8\pi}}\mathfrak{a}_2(u)\Big)^t \tag{3.12}$$
$$= -\Big(G_{1/\zeta}(\cdot;\lambda), G_{1/\zeta^2}(\cdot;\lambda), G_{1/\bar{\zeta}}(\cdot;\lambda), G_{1/\bar{\zeta}^2}(\cdot;\lambda)\Big)\, T^{-1}(\lambda)\times$$
$$\Big(\frac{1}{4\pi}\Big\langle f, G_{1/\bar{\zeta}}(\cdot,\bar{\lambda})\Big\rangle, \frac{1}{8\pi}\Big\langle f, G_{1/\bar{\zeta}^2}(\cdot,\bar{\lambda})\Big\rangle, \frac{1}{4\pi}\left\langle f, G_{1/\zeta}(\cdot;\bar{\lambda})\right\rangle, \frac{1}{8\pi}\left\langle f, G_{1/\zeta^2}(\cdot;\bar{\lambda})\right\rangle\Big)^t$$

(here, the $\times$ is just the usual matrix multiplication). Relation (3.2) immediately follows from (3.12), the elementary relation

$$\mathrm{Trace}\, g\,\langle\cdot, h\rangle = \langle g, h\rangle\,, \tag{3.13}$$

and the identities (3.11).

Similarly,

$$[(\Delta_{hol}-\lambda)^{-1}-(\Delta_F-\lambda)^{-1}](f) = \sum_{\alpha=1/\zeta^2,\ 1/\zeta} G_\alpha(\cdot;\lambda)X_\alpha(\lambda) \tag{3.14}$$

and

$$-\Big(\frac{1}{\sqrt{4\pi}}\mathfrak{a}_1(u), \frac{1}{\sqrt{8\pi}}\mathfrak{a}_2(u)\Big)^t = P(\lambda)\,(X_{1/\zeta}(\lambda), X_{1/\zeta^2}(\lambda))^t \tag{3.15}$$

with

$$P(\lambda) = \begin{pmatrix} S^{1/\zeta,\ \bar{\zeta}}(\lambda) & S^{1/\zeta^2,\ \bar{\zeta}}(\lambda) \\ S^{1/\zeta,\ \bar{\zeta}^2}(\lambda) & S^{1/\zeta^2,\ \bar{\zeta}^2}(\lambda) \end{pmatrix}, \tag{3.16}$$

and (3.3) follows from the same considerations as above. □

The next proposition is an immediate corollary of (2.21) (cf. Section 2.3.2).

Proposition 3.2. *Introduce the function $H(\cdot,\cdot)$ (both arguments are distinguished local parameters in a small vicinity of P) via*

$$W = \left[\frac{1}{(\zeta(Q)-\zeta(R))^2} + H(\zeta(Q),\zeta(R))\right] d\zeta(Q)d\zeta(R) \tag{3.17}$$

as $Q,R \to P$, where W is the canonical meromorphic bidifferential on X (in particular, one has the relation

$$6H(\zeta(P),\zeta(P)) = S_B(\zeta(P))\,,$$

where S_B is the Bergman projective connection). Then the matrix $T(0)$ is given via

$$\begin{pmatrix} T_{11}(0) \\ T_{21}(0) \\ T_{31}(0) \\ T_{41}(0) \end{pmatrix} = \begin{pmatrix} -\frac{1}{6}S_{Sch}(0) \\ -\frac{1}{2}H'_{\zeta'}(\zeta,\zeta')|_{(0,0)} + \frac{\pi}{2}\sum(\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta}v_\alpha(0)v'_\beta(0) \\ \pi B(0,0) \\ \frac{\pi}{2}\sum(\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta}v_\alpha(0)\overline{v'_\beta(0)} \end{pmatrix}, \tag{3.18}$$

$$\begin{pmatrix} T_{12}(0) \\ T_{22}(0) \\ T_{32}(0) \\ T_{42}(0) \end{pmatrix} = \begin{pmatrix} -H'_{\zeta}(\zeta,\zeta')|_{(0,0)} + \pi\sum(\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta}v_\alpha(0)v'_\beta(0) \\ -\frac{1}{2}H''_{\zeta\zeta'}(\zeta,\zeta')|_{(0,0)} + \frac{\pi}{2}\sum(\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta}v'_\alpha(0)v'_\beta(0) \\ \pi\sum(\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta}v'_\alpha(0)\overline{v_\beta(0)} \\ \frac{\pi}{2}\sum(\mathrm{Im}\,\mathbb{B})^{-1}_{\alpha\beta}v'_\alpha(0)\overline{v'_\beta(0)} \end{pmatrix}, \tag{3.19}$$

and

$$\begin{pmatrix} T_{13}(0) & T_{14}(0) \\ T_{23}(0) & T_{24}(0) \\ T_{33}(0) & T_{34}(0) \\ T_{43}(0) & T_{44}(0) \end{pmatrix} = \begin{pmatrix} \overline{T_{31}(0)} & \overline{T_{32}(0)} \\ \overline{T_{41}(0)} & \overline{T_{42}(0)} \\ \overline{T_{11}(0)} & \overline{T_{12}(0)} \\ \overline{T_{21}(0)} & \overline{T_{22}(0)} \end{pmatrix}. \tag{3.20}$$

One also has

$$P(0) = \begin{pmatrix} \pi B(0,0) & \pi \sum (\operatorname{Im} \mathbb{B})^{-1}_{\alpha\beta} v'_\alpha(0) \overline{v_\beta(0)} \\ \frac{\pi}{2} \sum (\operatorname{Im} \mathbb{B})^{-1}_{\alpha\beta} v_\alpha(0) \overline{v'_\beta(0)} & \frac{\pi}{2} \sum (\operatorname{Im} \mathbb{B})^{-1}_{\alpha\beta} v'_\alpha(0) \overline{v'_\beta(0)} \end{pmatrix}. \tag{3.21}$$

The following proposition describes the asymptotic behaviour of the S-matrix as $\lambda \to -\infty$.

Proposition 3.3. *All the entries of the matrix $T(\lambda)$ except $S^{1/\zeta,\bar\zeta}(\lambda)$, $S^{1/\zeta^2,\bar\zeta^2}(\lambda)$ and their conjugates $S^{1/\bar\zeta,\zeta}(\lambda)$, $S^{1/\bar\zeta^2,\zeta^2}(\lambda)$ are $O(|\lambda|^{-\infty})$ as $\lambda \to -\infty$. One has the asymptotics*

$$\begin{aligned} S^{1/\zeta,\bar\zeta}(\lambda) &= -\frac{2^{1/3}\sqrt{3}\Gamma(2/3)}{\pi\Gamma(4/3)}(-\lambda)^{1/3} + O(|\lambda|^{-\infty}); \\ S^{1/\zeta^2,\bar\zeta^2}(\lambda) &= -\frac{2^{-1/3}\sqrt{3}\Gamma(1/3)}{\pi\Gamma(5/3)}(-\lambda)^{2/3} + O(|\lambda|^{-\infty}); \end{aligned} \tag{3.22}$$

and

$$\det T(\lambda) = \left(\frac{27}{2\pi^2}\right)^2 \lambda^2 + O(|\lambda|^{-\infty}) \quad \text{and} \quad \det P(\lambda) = -\frac{27}{2\pi^2}\lambda + O(|\lambda|^{-\infty}) \tag{3.23}$$

as $\lambda \to -\infty$.

Proof. (cf. [12]). Passing to polar coordinates, r, ϕ such that $\zeta = r^{1/3}e^{i\phi/3}$; $0 \le \phi \le 6\pi$, one finds that the functions

$$K_\nu(\sqrt{-\lambda}r)e^{-i\nu\phi}; \quad \nu = \frac{1}{3}, \frac{2}{3},$$

where K_ν is the modified Bessel function (see discussion in Section 4.1.1), satisfy the equation (2.11) in a vicinity of P. The well-known asymptotics of the modified Bessel function (with $\nu > 0$) reads as

$$K_\nu(y) = \frac{\pi}{2\sin(\nu\pi)}\left[\frac{y^{-\nu}}{2^{-\nu}\Gamma(1-\nu)} - \frac{y^{\nu}}{2^{\nu}\Gamma(1+\nu)} + O(y^{2-\nu})\right]$$

as $y \to 0$. Thus, the functions $\Phi_\nu := \pi^{-1}2^{-\nu}\Gamma(1-\nu)\sin(\pi\nu)(\sqrt{-\lambda})^\nu K_\nu(\sqrt{-\lambda}r)e^{-i\nu\phi}$; $\nu = 1/3, 2/3$ satisfy (2.11) in a vicinity of P and have the asymptotics

$$\begin{aligned}\Phi_{1/3}(\zeta,\bar\zeta;\lambda) &= \frac{1}{\zeta} - \frac{2^{1/3}\sqrt{3}\Gamma(2/3)}{\pi\Gamma(4/3)}(-\lambda)^{1/3}\bar\zeta + o(|\zeta|^2)\\ \Phi_{2/3}(\zeta,\bar\zeta;\lambda) &= \frac{1}{\zeta^2} - \frac{2^{-1/3}\sqrt{3}\Gamma(1/3)}{\pi\Gamma(5/3)}(-\lambda)^{2/3}\bar\zeta^2 + o(|\zeta|^2)\end{aligned} \tag{3.24}$$

as $\zeta \to 0$.

Now, notice that one can change the construction of the special growing solutions from the proof of Proposition 2.5 replacing the function F by Φ_ν; this gives

$$\begin{aligned}G_{1/\zeta}(\cdot;\lambda) &= \Phi_{1/3}(\cdot;\lambda) - (\Delta_F-\lambda)^{-1}(\Delta^*-\lambda)[\chi\Phi_{1/3}(\cdot;\lambda)];\\ G_{1/\zeta^2}(\cdot;\lambda) &= \Phi_{2/3}(\cdot;\lambda) - (\Delta_F-\lambda)^{-1}(\Delta^*-\lambda)[\chi\Phi_{2/3}(\cdot;\lambda)]\,.\end{aligned} \tag{3.25}$$

Since $K_\nu(x)$ and all its derivatives are $O(e^{-x})$ as $x \to +\infty$ and the support of $(\Delta^*-\lambda)[\chi\Phi_\nu(\cdot;\lambda)]$ is separated from the origin, all the coefficients in the asymptotic expansions (2.7) of second terms in the right-hand sides of (3.25) are exponentially decreasing as $\lambda \to -\infty$ and, therefore, all the statements of the proposition follow from (3.24). □

The next proposition is a direct consequence of Theorem 2 from [13] and (3.23). The proof is provided in Section 4.4.

Proposition 3.4. *Introduce the zeta-regularized determinants of the operators* $\Delta_F - \lambda$*,*

$\Delta_{sing} - \lambda$, and $\Delta_{hol} - \lambda$ via

$$\det A = \exp\{-\zeta_A'(0)\},$$

where $\zeta_A(s)$ is the operator zeta-function of an operator A (without zero modes). Then

$$\det(\Delta_{sing} - \lambda) = \left(\frac{2\pi^2}{27}\right)^2 \det T(\lambda)\, \det(\Delta_F - \lambda) \tag{3.26}$$

for real λ not belonging to the union of the spectra of Δ_F and Δ_{sing}. Similarly,

$$\det(\Delta_{hol} - \lambda) = \frac{2\pi^2}{27} \det P(\lambda)\, \det(\Delta_F - \lambda) \tag{3.27}$$

for real λ not belonging to the union of the spectra of the operators Δ_F and Δ_{hol}.

Since $\dim\ker\Delta_F = 1$, the preceding proposition shows that the order of the zero of det $T(\lambda)$ (respectively, det $P(\lambda)$) at $\lambda = 0$ is one unit less than the dimension of the kernel of Δ_{sing} (respectively, Δ_{hol}). We shall prove in Section 3.2 that generically $\dim\ker\Delta_{hol} = 1$. We conjecture that this is also the case for Δ_{sing} (i.e., generically det $T(0) \neq 0$). However, we shall show that by choosing a "very symmetric" polyhedron X, one can get $\dim\ker\Delta_{sing} = 3$.

So, under assumption of genericity, passing to the limit $\lambda \to 0$ in (3.26) and (3.27), one gets the following comparison formulas for modified (i.e., with zero modes excluded) determinants of self-adjoint extensions Δ_F, Δ_{sing}, and Δ_{hol}.

Theorem 3.5. *Suppose* $\dim\ker\Delta_{sing} = 1$*. Then*

$$\det{}^*\Delta_{sing} = \left(\frac{2\pi^2}{27}\right)^2 \det T(0)\, \det{}^*\Delta_F, \tag{3.28}$$

where $T(0)$ is explicitly given by (3.18)-(3.20)*.*

Suppose $\dim\ker\Delta_{hol} = 1$ *(i.e., P is not a Weierstrass point of X, see Proposition*

3.10 below). Then

$$\det{}^* \Delta_{hol} = \frac{2\pi^2}{27} \det\, P(0)\, \det{}^* \Delta_F, \tag{3.29}$$

where $P(0)$ is given by (3.21).

Remark 3.6. *If $(2P) = C$, where C is in the canonical class, then the flat metric on X with a single conical point at P has the form $|\omega|^2$, where ω is a holomorphic differential on X with double zero at P. In this case, an explicit expression for $\det^* \Delta_F$ can be found in [22]. An explicit formula for $\det^* \Delta_F$ for an arbitrary P can be found in [18].*

Remark 3.7. *Let us mention two geometric constructions leading to a flat surface X of genus two with a single conical singularity.*

1. Take a compact Riemann surface X of genus two and choose a point $P \in X$. Then according to the Troyanov theorem (see [41]), there exists the unique (up to rescaling) flat conformal metric on X with conical singularity of angle 6π at P. If the divisor $(2P)$ is in the canonical class, then there exists a holomorphic one form ω on X with divisor $(2P)$ and the Troyanov metric necessarily coincides (up to resacaling) with $|\omega|^2$. In this case, the metric has trivial holonomy. If the divisor $(2P)$ does not belong to the canonical class, then the Troyanov metric must have nontrivial holonomy along some nontrivial cycle on X. (It should be noted that the holonomy of the Troyanov metric along a small loop around the conical point is always trivial: the tangent vector turns to the angle 6π after parallel transform along this loop.)

2. (See Figure 3.1.) In case of trivial holonomy, the flat surface X can be produced via the well known pentagon construction (see, e.g., [29]). Consider a pentagon Π

in the complex plane. Let the center of one of its sides coincide with the origin. Gluing the parallel sides of the octagon $\Pi \cup (-\Pi)$ together one gets a flat surface X of genus 2 with a single conical singularity of conical angle 6π. The one form dz in the complex plane gives rise to a holomorphic one form ω on X with a single double zero at the point P on X that came from eight vertices of the octagon glued together. The natural flat metric on X has trivial holonomy and coincides with $|\omega|^2$.

Now take the octagon $\Pi \cup (-\Pi)$ and deform it keeping the lengths of all the sides fixed (after this deformation the opposite sides are no longer parallel). Glue the sides together following the same gluing scheme as before. Again one gets a flat surface of genus two with a single conical singularity of angle 6π but now the corresponding flat metric has nontrivial holonomy: the parallel transport along the closed loop which came from a segment connecting two points on the opposite sides of the deformed octagon turns the tangent vector for the angle which is equal to the angle between these two opposite sides.

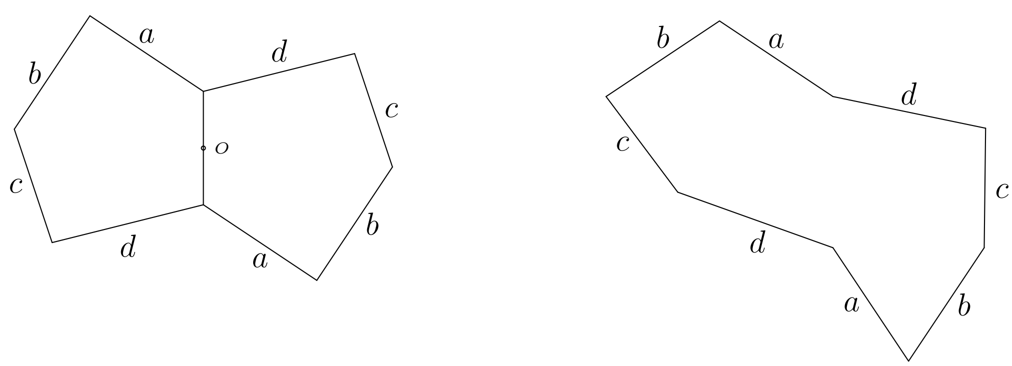

Figure 3.1: Gluing schemes for X: trivial (left) and nontrivial (right) holonomy

One more comparison formula for resolvent kernels.

Here we briefly describe an interesting counterpart to formula (3.14) which holds in case of general conformal flat conical metrics of trivial holonomy on compact Riemann surfaces X of genus $g \geq 2$. All these metrics have the form $|\omega|^2$, where ω is a holomorphic one form on X. Flat surfaces X of genus 2 with a single conical point P of angle 6π enter this class if and only if P is a Weierstrass point of X.

Proposition 3.8. *Let the metric on X be given by $|\omega|^2$, where ω is a holomorphic one form. Let $P_1, P_2, \dots, P_M$, $M \leq 2g-2$, be the distinct zeros of ω or, what is the same, the conical points of the metric $|\omega|^2$. Then there is the following relation between the resolvent kernels, R_{hol} and R_F, of the holomorphic and Friedrichs extensions of the symmetric Laplacian on $X \setminus \{P_1, \dots, P_M\}$:*

$$R_{hol}(x, y; \lambda) = \frac{4}{\lambda} \frac{1}{\omega(x)\overline{\omega(y)}} \partial_x \partial_{\bar{y}} R_F(x, y; \lambda) \tag{3.30}$$

Proof. We start with reminding the reader the standard relation

$$\partial_x \partial_{\bar{y}} G_F(x, y) = -\frac{1}{4} \sum_{\alpha,\beta=1}^{g} (\operatorname{Im} \mathbb{B})^{-1}_{\alpha\beta} v_\alpha(x) \overline{v_\beta(y)} = -\frac{1}{4} B(x, \bar{y}), \tag{3.31}$$

where $B(x, \bar{y})$ is the reproducing kernel for holomorphic differentials. Here G_F is just the Green function from (2.4), the subscript is introduced to emphasize that we deal with the Green function of the Friedrichs Laplacian. Equation (3.31) directly follows from (2.4) (the factor 1/4 appears due to the presence of the factor 4 in the definition of the Laplacian, some authors do not introduce these factors).

According to [11], one has the relations

$$\Delta_F = 4D_z^* D_z \quad \text{and} \quad \Delta_{hol} = 4D_z D_z^*, \tag{3.32}$$

where D_z is the closure of the operator

$$\frac{1}{\omega}\partial_z : C_0^\infty(X \setminus \{P_1, \dots, P_M\}) \subset L^2(X, |\omega|^2) \longrightarrow L^2(X, |\omega|^2)\,.$$

Clearly, D_z^* acts as $\frac{1}{\bar\omega}\partial_{\bar z}$.

Now (3.32) immediately implies that the function ϕ_m is a normalized eigenfunction of Δ_F corresponding to a nonzero eigenvalue λ_m if and only if $\frac{2}{\sqrt{\lambda_m}} D_z\phi_m$ is a normalized eigenfunction of Δ_{hol} corresponding the eigenvalue λ_m. Taking into account that $\ker\Delta_{hol}$ is spanned by the functions $\frac{v_\alpha}{\omega}$ and, therefore, the orthogonal projection in $L^2(X, |\omega|^2)$ onto $\ker\Delta_{hol}$ is the integral operator with the integral kernel $\dfrac{B(x,\bar y)}{\omega(x)\overline{\omega(y)}}$, one gets the following representation for the resolvent kernel of Δ_{hol} (in the sense of distribution theory):

$$R_{hol}(x, y; \lambda) = -\frac{B(x,\bar y)}{\omega(x)\overline{\omega(y)}}\frac{1}{\lambda} + 4\sum_{\lambda_m \neq 0} \frac{D_x\phi_m(x)\overline{D_y\phi_m(y)}}{(\lambda_m - \lambda)\lambda_m}\,. \tag{3.33}$$

Taking into account the relations

$$\frac{1}{(\lambda_m - \lambda)\lambda_m} = \frac{1}{\lambda}\left(\frac{1}{\lambda_m - \lambda} - \frac{1}{\lambda_m}\right),$$

$$R_F(x, y, \lambda) = -\frac{1}{\mathrm{Area}(X)\lambda} + \sum_{\lambda_m \neq 0} \frac{\phi_m(x)\overline{\phi_m(y)}}{\lambda_m - \lambda}\,,$$

and

$$G_F(x, y) = \sum_{\lambda_m \neq 0} \frac{\phi_m(x)\overline{\phi_m(y)}}{\lambda_m},$$

and making use of (3.31), one arrives at (3.30). □

Corollary 3.9. *For flat metrics $|\omega|^2$ with trivial holonomy, the Green function G_{hol} of* *the holomorphic extension Δ_{hol} is related to the Friedrichs Green function G_F via*

$$G_{hol}(x,y) = \int_X \partial_x G_F(x,z)\partial_{\bar{y}} G_F(z,y)\frac{1}{\omega(x)\overline{\omega(y)}}dS(z)\,.$$

3.2 Kernels of Δ_{hol} and Δ_{sing}.

3.2.1 Kernel of the holomorphic extension

The following proposition gives the complete description of the kernel of the holomorphic extension of the symmetric Laplacian on $X \setminus \{P\}$.

Proposition 3.10. *If P is not a Weierstrass point of X, then the kernel of Δ_{hol} consists of constants and so* $\dim\ker\,\Delta_{hol} = 1$.

If P is a Weierstrass point, then the kernel of Δ_{hol} has dimension 2*, and is spanned by* 1 *and a meromorphic function with single pole at P of multiplicity 2.*

Proof. Let $u \in \ker\,\Delta_{hol}$. Let ζ be the distinguished local parameter near P and let $X_\epsilon = X \setminus \{|\zeta| \leq \epsilon\}$. Using Stokes formula, one gets

$$0 = -\frac{1}{4}\langle u, \Delta u\rangle \quad = \lim_{\epsilon\to 0}\Big\{ \iint_{X_\epsilon} |\bar\partial u|^2 + \oint_{|\zeta|=\epsilon} \overline{(A/\zeta^2 + B/\zeta + C + D\zeta + E\zeta^2 + o(|\zeta|^2))}\times$$

$$\partial_{\bar\zeta}(A/\zeta^2 + B/\zeta + C + D\zeta + E\zeta^2 + o(|\zeta|^2))\Big\} = \iint_X |\bar\partial u|^2$$

and, therefore, u is meromorphic on X with a single pole of degree less or equal to 2 at P. It remains to notice that

- there are no meromorphic functions with a single pole of order 1 on Riemann surfaces of positive genus; and

- for Riemann surfaces X of genus 2, the point $P \in X$ is a Weierstrass point if and only if there exists a meromorphic function on X with single double pole at P.

The proof is complete. □

Remark 3.11. *If P is a Weierstrass point (and, therefore,* $\dim\ker\Delta_{hol} = 2$*), then*

$$\det{}^*\Delta_{hol} = \frac{2\pi^2}{27}\frac{d\det P(\lambda)}{d\lambda}\Big|_{\lambda=0}\det{}^*\Delta_F. \tag{3.34}$$

The value of the factor $\frac{d\det P(\lambda)}{d\lambda}\Big|_{\lambda=0}$ in (3.34) can be obtained by explicitly calculating the derivatives of the entries of $P(\lambda)$ at $\lambda = 0$. The computation is similar to the one after Remark 3.13 below so we skip the details here.

3.2.2 Singular extension: very symmetric case

Consider a hyperelliptic surface X of genus 2 via $\mu^2 = \prod_{j=1}^{6}(\lambda - \lambda_j)$ with $\lambda_k = \lambda_1 + r^2 e^{\frac{2\pi i(k-1)}{5}}$; $k = 2, \dots, 6$; $r > 0$. Consider a holomorphic 1-form ω on X given by

$$\omega = \frac{(\lambda - \lambda_1)\, d\lambda}{\sqrt{\prod_{j=1}^{6}(\lambda - \lambda_j)}}.$$

Clearly, ω has a double zero at $P = (\lambda_1, 0) \in X$ and the metric $|\omega|^2$ is a flat metric on X with unique conical point at P of angle 6π.

Proposition 3.12. *The kernel of the singular self-adjoint extension Δ_{sing} of a symmetric Laplacian on $X \setminus \{P\}$ has dimension* 3.

Proof. There are two natural holomorphic local parameters on X near P: the one related to the ramified double covering $X \ni (\lambda, \mu) \mapsto \lambda \in \mathbb{C} \subset \mathbb{P}^1$,

$$\zeta = \sqrt{\lambda - \lambda_1}\,,$$

and the distinguished local parameter ξ for the conical metric $|\omega|^2$ related to the parameter ζ via

$$\xi(\zeta) = \left(\int_0^{\zeta} \frac{2w^2 dw}{\sqrt{w^{10} - r^{10}}} \right)^{1/3}.$$

Since

$$\omega = C(\zeta^2 + O\left(\zeta^{12}\right))\, d\zeta$$

and, therefore,

$$\xi^3 = C\left(\zeta^3 + O(\zeta^{13})\right), \tag{3.35}$$

one has

$$\frac{1}{\zeta^2} = \frac{C}{\xi^2} + o(|\xi|^3) \tag{3.36}$$

as $\xi, \zeta \to 0$ (the constant C differs from one formula to another). Now, equation (3.36) implies that the meromorphic function

$$P \mapsto f(P) = \frac{1}{\lambda(P) - \lambda_1} \tag{3.37}$$

on X with only one double pole at P belongs to $\ker \Delta_{sing}$. Clearly, the complex conjugate $\bar{f}$ and 1 also belong to $\ker \Delta_{sing}$. Thus, $\dim \ker \Delta_{sing} \geq 3$.

It turns out that in the case of the surface X, one can further specify the asymptotic expansion of the (unique up to a constant) harmonic function g on X with a single singularity at P with

$$g(\xi,\bar{\xi})=\frac{1}{\xi}+O(1)\,. \tag{3.38}$$

Namely, one has

$$g=\frac{1}{\xi}+C+\alpha\bar{\xi}+o(|\xi|^2) \tag{3.39}$$

with $\alpha\neq 0$. This means that in the asymptotic expansion of the function g, there are no ξ, ξ^2, and $\bar{\xi}^2$ terms. Indeed, according to (2.30) the coefficient near ξ in the asymptotic expansion of g near P is equal to $-\frac{1}{6}S_{Sch}(\xi)|_{\xi=0}$. Using the $\mathbb{Z}_5$-symmetry of X, it is easy to show that this coefficient must vanish. First, notice that this quantity vanishes if

$$S_{Sch}(\zeta)|_{\zeta=0}=0. \tag{3.40}$$

This follows from the change of variables rule for a projective connection:

$$S_{Sch}(\xi)=S_{Sch}(\zeta)\left(\frac{d\zeta}{d\xi}\right)^2+\{\zeta,\xi\} \tag{3.41}$$

(due to (3.35), the Schwarzian derivative in the right-hand side of the last equality vanishes at $\xi=0$).

Without loss of generality one can assume that $\lambda_1=0$. Consider the automorphism of X

$$\lambda\mapsto e^{\frac{2\pi i}{5}}\lambda\,.$$

Under the automorphism $\zeta\mapsto e^{\frac{\pi i}{5}}\zeta$ and, since the Schiffer projective connection is independent of the choice of basic cycles on X, one gets from (3.41) the relation

$$S_{Sch}(\zeta)|_{\zeta=0}=e^{\frac{2\pi i}{5}}S_{Sch}(\zeta)|_{\zeta=0},$$

which implies $S_{Sch}(\zeta)|_{\zeta=0}=0$, and, therefore, the term ξ is absent.

Next, using the matrix $T(0)$ (refer to equations (3.18)-(3.20)) and (3.37) together, one can easily show that there are no ξ^2 and $\bar{\xi}^2$ terms in the asymptotic expansion of g. Indeed, notice that the normalized holomorphic differentials v_1 and v_2 on X are linear combinations of

$$\omega_1 = \frac{d\lambda}{\sqrt{\prod_{k=1}^{6}(\lambda - \lambda_k)}} = \frac{2d\zeta}{\sqrt{\zeta^{10} - r^{10}}}$$

and

$$\omega_2 = \frac{\lambda d\lambda}{\sqrt{\prod_{k=1}^{6}(\lambda - \lambda_k)}} = \frac{2\zeta^2 d\zeta}{\sqrt{\zeta^{10} - r^{10}}},$$

and, therefore,

$$v'_{1,2}(\zeta)|_{\zeta=0} = 0\,.$$

Since

$$v'_{1,2}(\xi) = v'_{1,2}(\zeta)\left(\frac{d\zeta}{d\xi}\right)^2 + v_{1,2}(\zeta)\frac{d^2\zeta}{d\xi^2}$$

and $\left.\frac{d^2\zeta}{d\xi^2}\right|_{\xi=0} = 0$ (follows immediately from (3.35)), one gets

$$v'_{1,2}(\xi)|_{\xi=0} = 0\,. \tag{3.42}$$

Relation (3.37) implies that one has $T_{12}(0) = 0$ in (3.19), and from the symmetry $H(x,y) = H(y,x)$ of the function H from Proposition 3.2 and (3.42), one concludes that $T_{21}(0) = 0$. Therefore, there is no ξ^2 term in the expansion of g. Due to (3.42), one has $T_{41}(0) = 0$, and, therefore, the term $\bar{\xi}^2$ is also absent.

It remains to notice that the coefficient α of the term $\bar{\xi}$ equals to $\pi B(\xi, \bar{\xi})|_{\xi=0}$. Since the imaginary part of the matrix of b-periods, $\mathrm{Im}\,\mathbb{B}$, is positive definite, one has $\alpha \neq 0$, and (3.39) is proved.

To prove that $\ker \Delta_{sing} = \text{lin.span}\{f, \bar{f}, 1\}$, it suffices to prove that a function W from $\mathcal{D}(\Delta^*)$ with asymptotics

$$W = \frac{A}{\xi} + \frac{B}{\bar{\xi}} + o(|\xi|^2)$$

cannot belong to $\ker \Delta_{sing}$, unless $A = B = 0$. Assuming $W \in \ker \Delta_{sing}$, one finds that $W - Ag - B\bar{g} \in \ker \Delta_F$, and, therefore,

$$W = Ag + B\bar{g} + C$$

which contradicts (3.39), unless $A = B = 0$. This completes the proof. □

Remark 3.13. *In the case of the very symmetric surface X with $\dim \ker \Delta_{sing} = 3$, the comparison formula for the determinants (i.e., equation (3.28)) turns into*

$$\det{}^* \Delta_{sing} = 2 \left(\frac{\pi^2}{27}\right)^2 \frac{d^2 \det T(\lambda)}{d\lambda^2}\Big|_{\lambda=0} \det{}^* \Delta_F. \tag{3.43}$$

It should be noticed that the derivatives of the entries of the matrix $T(\lambda)$ from (3.6) at $\lambda = 0$ (and, therefore, the factor $\frac{d^2 \det T(\lambda)}{d\lambda^2}\Big|_{\lambda=0}$ in (3.43)) can be explicitly computed. Namely, explicit expressions for the derivatives of the first order can be obtained via plugging $\lambda = 0$ in (3.11) and then using (2.21). To get expressions for the second derivatives, for instance $\frac{d^2}{d\lambda^2} S^{1/\zeta^2,\, \zeta}(\lambda)\Big|_{\lambda=0}$, introduce (following the proof of Proposition 2.5) $F = \chi \frac{1}{\zeta^2}$ and the solution $g(\,\cdot\,; \lambda)$ of the equation

$$(\Delta_F - \lambda) g = (\Delta^* - \lambda) F\,. \tag{3.44}$$

Then $G_{1/\zeta^2}(\,\cdot\,; \lambda) = F - g(\,\cdot\,, \lambda)$. Denoting by dot the derivative with respect to λ and differentiating (3.44) (cf. Section 4.5), one gets

$$-\dot{g} = (\Delta_F - \lambda)^{-1} G_{1/\zeta^2}(\,\cdot\,; \lambda) \tag{3.45}$$

and

$$(\Delta_F - \lambda)(-\ddot{g}) = 2(-\dot{g})\,. \tag{3.46}$$

Now, (3.8) gives

$$\begin{aligned}\sqrt{4\pi}\frac{d^2}{d\lambda^2}S^{1/\zeta^2,\zeta}(\lambda)\Big|_{\lambda=0} &= \left\langle 2(-\dot{g}), G_{1/\zeta}(\,\cdot\,;\bar{\lambda})\right\rangle \\ &= 2\left\langle(\Delta_F - \lambda)^{-1}G_{1/\zeta^2}(\,\cdot\,;\lambda), G_{1/\zeta}(\,\cdot\,;\bar{\lambda})\right\rangle\,.\end{aligned}$$

Since $G_{1/\zeta^2}(\,\cdot\,;0)\perp 1$, this implies

$$\frac{d^2}{d\lambda^2}S^{1/\zeta^2,\zeta}(\lambda)\Big|_{\lambda=0} = \frac{1}{\sqrt{\pi}}\int_X\int_X G(x,y)G_{1/\zeta^2}(x;0)G_{1/\bar{\zeta}}(y;0)dS(y)dS(x),$$

where $G(x,y)$ is the Green function from (2.4) and the special growing solutions $G_{1/\zeta^2}(\,\cdot\,;0)$ and $G_{1/\bar{\zeta}}(\,\cdot\,;0)$ are explicitly computed in (2.21).

Chapter 4

Proofs of auxiliary results from Chapters 2 and 3

In this chapter, we provide the proofs of several results from the main chapters.

4.1 Proof of Proposition 2.1

The proof of the following proposition from Chapter 2 was outlined in the appendix of [19]. In this section, the details of the proof are presented. An alternative proof (of a closely related statement) based on different technical tools can be found in [30].

Proposition 2.1. *In the vicinity of the point P_j, a function $u \in \mathcal{D}(\Delta^*)$ has the asymptotics*

$$\begin{aligned} u \;&= \frac{i}{\sqrt{2\pi}}\mathfrak{L}_j(u)\log|\zeta_j| + \sum_{m=1}^{n_j}\frac{1}{\sqrt{4\pi m}}\mathfrak{H}_{j,m}(u)\frac{1}{\zeta_j^m} + \sum_{m=1}^{n_j}\frac{1}{\sqrt{4\pi m}}\mathfrak{A}_{j,m}(u)\frac{1}{\bar{\zeta}_j^m} \\ &+\frac{i}{\sqrt{2\pi}}\mathfrak{c}_j(u) + \sum_{m=1}^{n_j}\frac{1}{\sqrt{4\pi m}}\mathfrak{h}_{j,m}(u)\zeta_j^m + \sum_{m=1}^{n_j}\frac{1}{\sqrt{4\pi m}}\mathfrak{a}_{j,m}(u)\bar{\zeta}_j^m + \chi v, \end{aligned} \tag{2.7}$$

where χ is a smooth cut-off function that has compact support in a small vicinity of P_j and that is equal to 1 *in a smaller vicinity, and v is a function from the domain of the closure $\mathcal{D}\left(\overline{\Delta}\right)$. One has the asymptotics $v = o(|\zeta_j|^{n_j})$ as $\zeta_j \to 0$.*

From here and until the end of the section, we shall remove the subscript j appearing in the proposition. So, let P be a conical point with conical angle $\beta = 2\pi(b+1)$. Let n be the integer such that $2\pi n \ < \beta \leq 2\pi(n+1)$ and denote by ζ the distinguished local parameter near P. Recall that the part of the Riemann surface X near any conical point is isometric to a neighborhood of the tip of a Euclidean cone (see, e.g., [11]). For the given conical point P, denote by $\mathcal{K}$ the cone in $\mathbb{R}^2$ with vertex at $\mathcal{O} = \zeta(P)$. In polar coordinates, $\mathcal{K}$ has the representation

$$\mathcal{K} = \{(r,\theta) : r > 0, \theta \in \mathbb{S}^1_\beta := [0,\beta]\}. \tag{4.1}$$

Introduce the Sobolev space $H^l(\mathcal{K})$ with norm

$$\left\|u; H^l(\mathcal{K})\right\| = \left(\int_{\mathcal{K}} \sum_{|\alpha|\leq l} |\partial_x^\alpha u(x)|^2\, dx\right)^{1/2}. \tag{4.2}$$

In particular, $H^0(\mathcal{K}) := L^2(\mathcal{K})$. Introduce also the weighted Sobolev space $H^l_\gamma(\mathcal{K})$ with norm

$$\left\|u; H^l_\gamma(\mathcal{K})\right\| = \left(\sum_{|\alpha|\leq l} \int_{\mathcal{K}} r^{2(\gamma-l+|\alpha|)} |\partial_x^\alpha u(x)|^2\, dx\right)^{1/2}. \tag{4.3}$$

This norm is equivalent to the norm

$$\Big(\sum_{j=0}^{l}\sum_{k=0}^{l-j}\int_0^\beta\int_0^\infty r^{2(\nu+j)-1}|(r\partial_r)^j\partial_\theta^k\, u(r,\theta)|^2\, dr\, d\theta\Big)^{1/2},$$

where $\gamma = \nu + l - 1$. Observe that if $u \in H^l_\gamma(\mathcal{K})$, then $r^\gamma u \in H^l(\mathcal{K})$.

We remind the readers about the definition of Mellin transformation and some of its basic properties. For a given function $u \in C_0^\infty(\mathbb{R}^+)$, its *Mellin transformation* is given by

$$\widehat{u}(\lambda) = (\mathcal{M}_{r\to\lambda} u)(\lambda) = \int_0^\infty r^{-\lambda-1} u(r) dr. \tag{4.4}$$

Lemma 4.1 (See Lemma 6.1.3 in [23])**.**

1. *The Mellin transformation is a linear and continuous mapping from $C_0^\infty(\mathbb{R}^+)$ into the space of analytic functions on $\mathbb{C}$.*

2. *Every function $u \in C_0^\infty(\mathbb{R}^+)$ satisfies*

$$\mathcal{M}_{r\to\lambda}(r\partial_r\, u) = \lambda \mathcal{M}_{r\to\lambda} u.$$

Furthermore, for all $u, v \in C_0^\infty(\mathbb{R}^+)$, the Parseval equality

$$\int_0^\infty r^{2\gamma-1} u(r)\overline{v(r)}\, dr = \frac{1}{2\pi i} \int_{\mathrm{Re}\,\lambda=-\gamma} \widehat{u}(\lambda)\overline{\widehat{v}(\lambda)}\, d\lambda \tag{4.5}$$

is valid.

3. *The inverse Mellin transformation is given by*

$$u(r) = (\mathcal{M}^{-1}_{\lambda\to r}\widehat{u})(r) = \frac{1}{2\pi i} \int_{\mathrm{Re}\,\lambda=-\gamma} r^\lambda\, \widehat{u}(\lambda)\, d\lambda.$$

4. *If $r^{\gamma_i-1/2}u \in L^2(\mathbb{R}^+)$ for $i = 1, 2$, where $\gamma_1 < \gamma_2$ are arbitrary real numbers, then $\widehat{u}$ is holomorphic in the strip $-\gamma_2 < \mathrm{Re}(\lambda) < -\gamma_1$.*

4.1.1 Solutions to the homogeneous equation $(\Delta^* - \rho^2)u = 0$

Consider the self-adjoint Laplace operator $L = -(b+1)^{-2}\partial_\theta^2$ on $L^2(\mathbb{S}^1_\beta)$. It is easy to show that its eigenvalues μ_k are of the form $\mu_k = k^2/(b+1)^2$ with corresponding eigenfunctions $\varphi_k = e^{ik\theta}$, $k = 0, \pm 1, \pm 2, \ldots$.

Introduce the operator pencil

$$\mathscr{A}(\lambda) = L + (i\lambda)^2 = L - \lambda^2, \tag{4.6}$$

which is defined on the Sobolev space $H^2(\mathcal{K})$. One easily finds that the spectrum of the pencil consists of $\lambda_k = k/(b+1)$.

Consider the homogeneous problem

$$\left(\Delta^* - \rho^2\right) u = 0 \quad \text{in } \ \mathcal{K}, \tag{4.7}$$

where ρ is a complex parameter. For $k > 0$, let $\lambda_{-k} = -\sqrt{\mu_{-k}} = -k/(b+1)$. We shall find a solution u of the problem (4.7) such that $u \sim r^{\lambda_{-k}}\varphi_{-k}$ as $r \to 0$ and $u \in L^2(\mathcal{K}\backslash B_\epsilon)$, where $B_\epsilon := \{x \in \mathbb{R}^2 : |x| < \epsilon\}$ for sufficiently small $\epsilon > 0$. Set

$$u(r, \theta, \rho) = r^{\lambda_{-k}}\zeta(r\rho)\varphi_{-k}(\theta),$$

where ζ (not the distinguished local parameter) is some function to be determined and such that $\zeta(0) = 1$. Observe that

$$\Big((r\partial_r)^2 + (b+1)^{-2}\partial_\theta^2\Big)(r^{\lambda_{-k}}\varphi_{-k}(\theta)) = \lambda_{-k}^2 r^{\lambda_{-k}}\varphi_{-k}(\theta) - k^2(b+1)^{-2}r^{\lambda_{-k}}\varphi_{-k}(\theta) = 0.$$

Thus,

$$\begin{aligned}
0 \ &= -r^2\left(\Delta^* - \rho^2\right) u(r,\theta,\rho) = \Big((r\partial_r)^2 + (b+1)^{-2}\partial_\theta^2 + (r\rho)^2\Big)(r^{\lambda_{-k}}\zeta(r\rho)\varphi_{-k}(\theta)) \\
&= \varphi_{-k}(\theta)\Big[\lambda_{-k}^2 r^{\lambda_{-k}}\zeta(r\rho) + (2\lambda_{-k}+1)r^{\lambda_{-k}}(r\rho)\zeta'(r\rho) + r^{\lambda_{-k}}(r\rho)^2\zeta''(r\rho)\Big] \\
&\qquad -\lambda_{-k}^2 r^{\lambda_{-k}}\zeta(r\rho)\varphi_{-k}(\theta) + (r\rho)^2 r^{\lambda_{-k}}\varphi_{-k}(\theta)\zeta(r\rho) \\
&= \varphi_{-k}(\theta)r^{\lambda_{-k}}\Big((r\rho)^2\zeta(r\rho) + (2\lambda_{-k}+1)(r\rho)\zeta'(r\rho) + (r\rho)^2\zeta''(r\rho)\Big),
\end{aligned}$$

and, so,

$$(r\rho)^2\zeta(r\rho) + (2\lambda_{-k} + 1)(r\rho)\zeta'(r\rho) + (r\rho)^2\zeta''(r\rho) = 0. \tag{4.8}$$

Put $y = r\rho$ and $\zeta(y) = y^\nu \xi(y)$, where $\nu = -\lambda_{-k} = k/(b+1)$. It follows from (4.8) that

$$y^2\xi''(y) + y\xi'(y) + (y^2 - \nu^2)\xi(y) = 0. \tag{4.9}$$

Now, put $\Theta(y) = \xi(-iy)$. Then (4.9) turns to the modified Bessel's equation

$$y^2\Theta''(y) + y\Theta'(y) - (y^2 + \nu^2)\Theta(y) = 0. \tag{4.10}$$

Thus, taking $\Theta(y) = K_\nu(y)$, the modified Bessel's function of the second kind, one obtains

$$\zeta(r\rho) = c(r\rho)^\nu K_\nu(ir\rho),$$

where c is a constant satisfying the condition $\zeta(0) = 1$. In particular,

$$\begin{aligned} c^{-1}i^\nu &= \lim_{z\to 0}(iz)^\nu K_\nu(iz) = \lim_{z\to 0}\frac{\pi(iz)^\nu}{2\sin(\pi\nu)}\Big[I_{-\nu}(iz) - I_\nu(iz)\Big] \\ &= \lim_{z\to 0}\frac{\pi(iz)^\nu}{2\sin(\pi\nu)}\left[\sum_{m=0}^{\infty}\frac{(iz/2)^{2m-\nu}}{m!\Gamma(m-\nu+1)} - \sum_{m=0}^{\infty}\frac{(iz/2)^{2m+\nu}}{m!\Gamma(m+\nu+1)}\right] \\ &= \frac{\pi 2^{\nu-1}}{\sin(\pi\nu)\Gamma(1-\nu)}, \end{aligned}$$

and, therefore, $c = \pi^{-1}\sin(\pi\nu)\Gamma(1-\nu)i^\nu 2^{1-\nu}$. In the computation above, I_ν is the modified Bessel's function of the first kind and is given by

$$I_\nu(z) = \sum_{m=0}^{\infty}\frac{(z/2)^{2m+\nu}}{m!\Gamma(m+\nu+1)}. \tag{4.11}$$

Now, K_ν has the asymptotics (see, e.g., [1], equation 9.7.2 on page 378)

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} \exp(-z) \left[\sum_{m=0}^{N-1} c_{\nu,m}(2z)^{-m} + O(|z|^{-N})\right]$$

as $|z| \to +\infty$. Since $\exp(-z)$ decreases rapidly as $|z| \to +\infty$, it follows that the solution

$$u(r,\theta,\rho) = c(r\rho)^\nu r^{\lambda_{-k}} K_\nu(ir\rho)\varphi_{-k}(\theta)$$

belongs to $L^2(\mathcal{K} \setminus B_\epsilon)$. Thus, for $k > 0$, set

$$w_{-k}(r,\theta,\rho) = \frac{2^{1-\nu}}{\Gamma(\nu)}(ir\rho)^\nu r^{\lambda_{-k}} K_\nu(ir\rho)\varphi_{-k}(\theta), \tag{4.12}$$

where $\nu = k/(b+1)$.

Similarly, let $\lambda_k = k/(b+1)$, $k > 0$. We now find a solution u of (4.7) such that $u \sim r^{\lambda_k}\varphi_k$ as $r \to 0$. As in before, set $u(r,\theta,\rho) = r^{\lambda_k}\zeta(r\rho)\varphi_k(\theta)$, where $\zeta(r\rho)$ is a function to be determined and such that $\zeta(0) = 1$. With some modifications in the computations above, one gets

$$u(r,\theta,\rho) = 2^\nu\Gamma(1+\nu)(ir\rho)^{-\nu} I_\nu(ir\rho) r^{\lambda_k}\varphi_k(\theta),$$

where $\nu = k/(b+1)$. Hence, for $k > 0$, set

$$\begin{aligned} w_k(r,\theta,\rho) &= 2^\nu\Gamma(1+\nu)(ir\rho)^{-\nu} I_\nu(ir\rho) r^{\lambda_k}\varphi_k(\theta) \\ &= \Gamma(1+\nu) r^{\lambda_k}\varphi_k(\theta) \sum_{m=0}^{\infty} \frac{(ir\rho/2)^{2m}}{m!\Gamma(m+\nu+1)}. \end{aligned} \tag{4.13}$$

Finally, we shall find a solution u of (4.7) such that $u \sim \ln r$ as $r \to 0$. This solution corresponds to the eigenvalue $\lambda_0 = 0$ of the operator pencil $\mathscr{A}(\lambda)$. Since that eigenvalue λ_0 has algebraic multiplicity 2, one finds two linearly independent solutions, namely

$$w_{01}(r,\theta,\rho) = c_{1\beta} I_0(ir\rho) \quad \text{and} \quad w_{02}(r,\theta,\rho) = c_{2\beta} K_0(ir\rho), \tag{4.14}$$

where $c_{1\beta}$ and $c_{2\beta}$ depend only on the conical angle β. These solutions can be obtained by setting $u(r,\theta,\rho)=\zeta(r\rho)$. It follows that

$$0=-r^2(\Delta^*-\rho^2)\zeta(r\rho)=(r\rho)^2\zeta''(r\rho)+(r\rho)\zeta'(r\rho)+(r\rho)^2\zeta(r\rho).$$

Using a similar computation as in above, the last equations turns to (4.10), with $\nu=0$, whose two linearly independent solutions are of the form (4.14). Of course, $K_0(z)$ has a logarithmic singularity at $z\to 0$, while $I_0(0)$ is finite.

In summary, the solutions of the problem (4.7) are of the form

$$w_k(r,\theta,\rho)=\begin{cases}\dfrac{2^{1-\nu}}{\Gamma(\nu)}(ir\rho)^{\nu}K_\nu(ir\rho)\,r^{\lambda_k}\varphi_k(\theta), & \text{if } k<0\\[2ex] 2^{\nu}\Gamma(1+\nu)(ir\rho)^{-\nu}I_\nu(ir\rho)\,r^{\lambda_k}\varphi_k(\theta), & \text{if } k>0\\[2ex] c_{2\beta}K_0(ir\rho)+c_{1\beta}I_0(ir\rho), & \text{if } k=0.\end{cases} \tag{4.15}$$

4.1.2 Some a priori estimates

Consider the model problem

$$\Delta^*u=f \quad \text{in } \mathcal{K}, \tag{4.16}$$

where $f\in L^2(\mathcal{K})$. Writing in polar coordinates, equation (4.16) is equivalent to the problem

$$-\left((r\partial_r)^2+(b+1)^{-2}\partial_\theta^2\right)u(r,\theta)=r^2f(r,\theta)=:F(r,\theta). \tag{4.17}$$

If one applies Mellin transformation (4.4) to (4.17), the preceding equation turns to an ordinary differential equation with parameter problem

$$(L-\lambda^2)\widehat{u}(\lambda,\theta)=\widehat{F}(\lambda,\theta), \tag{4.18}$$

where $\widehat{F} \in L^2(\mathbb{S}^1_\beta)$ (compare the operator in (4.18) with the operator pencil (4.6)). If one finds solutions of (4.18) for every $\lambda \in \mathbb{C}$, using the inverse Mellin transformation on these solutions gives solutions of (4.17).

First, consider the Green function (the integral kernel of the inverse operator)

$$\Phi(|x-y|) = \frac{\pi}{\lambda} e^{-\lambda|x-y|}$$

of the operator $\lambda^2 - (d/dx)^2$ on $\mathbb{R}$ (see [40], equation (5.30) on page 220). Then the Green function of (4.18) is given by

$$\sum_{n\in\mathbb{Z}} \Phi(|\theta - \tau + \beta n|).$$

The sum of the preceding series gives the needed expression for the Green function of (4.18):

$$\Gamma(\theta,\tau;\lambda) = \frac{\pi}{\lambda}\,\frac{e^{-\lambda|\theta-\tau|} + e^{\lambda|\theta-\tau|}e^{-\beta}}{1-e^{-\beta\lambda}} = -\frac{\pi}{\lambda^2}\,\frac{e^{-\lambda|\theta-\tau|} + e^{\lambda|\theta-\tau|}e^{-\beta}}{\sum_{n=1}^{\infty}(-\beta)^n\lambda^{n-1}(n!)^{-1}}. \tag{4.19}$$

The last expression implies that the Green function Γ has a double pole at $\lambda = 0$ and simple poles at $\lambda_k = k/(b+1)$, $k = \pm1, \pm2, \ldots$. It follows that

$$\widehat{u}(\theta,\lambda) = \int_0^\beta \Gamma(\theta,\tau;\lambda)\widehat{F}(\tau,\lambda)\,d\tau.$$

Using Cauchy-Schwarz inequality, its $L^2(\mathbb{S}^1_\beta)$-norm satisfies

$$\begin{aligned}\left\|\widehat{u}(\cdot,\lambda); L^2(\mathbb{S}^1_\beta)\right\|^2 &= \int_0^\beta \left|\int_0^\beta \Gamma(\theta,\tau;\lambda)\widehat{F}(\tau,\lambda)\,d\tau\right|^2 d\theta \\ &\le \left\|\widehat{F}; L^2(\mathbb{S}^1_\beta)\right\|^2 \int_0^\beta\int_0^\beta |\Gamma(\theta,\tau;\lambda)|^2\,d\tau\,d\theta.\end{aligned}$$

Furthermore, the estimate

$$\int_0^\beta\int_0^\beta |\Gamma(\theta,\tau;\lambda)|^2\,d\tau\,d\theta \le c|\lambda|^{-4}$$

holds true for $\mathrm{Re}(\lambda) \neq k/(b+1)$. It follows that

$$\left\| \widehat{u}(\cdot,\lambda); L^2(\mathbb{S}^1_\beta) \right\|^2 \leq c_1 |\lambda|^{-4} \left\| \widehat{F}; L^2(\mathbb{S}^1_\beta) \right\|^2, \tag{4.20}$$

with $\mathrm{Re}(\lambda) \neq k/(b+1)$ and c_1 is independent of λ. Meanwhile,

$$\frac{\partial}{\partial \theta}\Gamma(\theta,\tau;\lambda) = \frac{\pi(\theta-\tau)}{|\theta-\tau|}\ \frac{-e^{-\lambda|\theta-\tau|} + e^{\lambda|\theta-\tau|}e^{-\beta}}{1-e^{-\beta\lambda}},$$

and, therefore, the estimate

$$\left\| \partial_\theta \widehat{u}(\cdot,\lambda); L^2(\mathbb{S}^1_\beta) \right\|^2 \leq c_2 |\lambda|^{-2} \left\| \widehat{F}; L^2(\mathbb{S}^1_\beta) \right\|^2 \tag{4.21}$$

holds for $\mathrm{Re}(\lambda) \neq k/(b+1)$ and c_2 is independent of λ. Also, using (4.18), one can write $\partial_\theta^2 \widehat{u} = -(b+1)^2\lambda^2\widehat{u} - (b+1)^2\widehat{F}$, and, hence,

$$\left\| \partial_\theta^2 \widehat{u}(\cdot,\lambda); L^2(\mathbb{S}^1_\beta) \right\| \leq (b+1)^2|\lambda|^2 \left\| \widehat{u}(\cdot,\lambda); L^2(\mathbb{S}^1_\beta) \right\| + (b+1)^2 \left\| \widehat{F}; L^2(\mathbb{S}^1_\beta) \right\|.$$

Thus, the last inequality together with (4.20) imply

$$\left\| \partial_\theta^2 \widehat{u}(\cdot,\lambda); L^2(\mathbb{S}^1_\beta) \right\|^2 \leq c_3 \left\| \widehat{F}; L^2(\mathbb{S}^1_\beta) \right\|^2. \tag{4.22}$$

Combining the estimates (4.20)-(4.22) yields to

$$\sum_{j=0}^{2} |\lambda|^{2j} \left\| \partial_\theta^{2-j} \widehat{u}(\cdot,\lambda); L^2(\mathbb{S}^1_\beta) \right\|^2 \leq C \left\| \widehat{F}; L^2(\mathbb{S}^1_\beta) \right\|^2 \tag{4.23}$$

for $\mathrm{Re}(\lambda) \neq k/(b+1)$, $k = 0, \pm1, \pm2, \dots$ and C is independent of λ. In particular, if b is not an integer (or, equivalently, the conical angle β is not a multiple of 2π), the inequality (4.23) remains valid when $|\lambda|$ is replaced by 1, and, therefore,

$$\sum_{j=0}^{2} (1+|\lambda|^2)^j \left\| \partial_\theta^{2-j} \widehat{u}(\cdot,\lambda); L^2(\mathbb{S}^1_\beta) \right\|^2 \leq C \left\| \widehat{F}; L^2(\mathbb{S}^1_\beta) \right\|^2 \tag{4.24}$$

still holds.

Now, let $0 < \delta < 1/2$ be sufficiently small and consider the tip of the cone $\mathcal{K}_\delta = \{(r,\theta) : 0 < r < 2\delta, \theta \in [0,\beta]\}$. Note that all the estimates above remain valid (extend all functions to whole of $\mathcal{K}$ by zero). Using Parseval's identity (4.5) with $\mathrm{Re}(\lambda) = -\gamma \neq k/(b+1)$, the left-hand side of (4.24) converts to

$$\begin{aligned}\int_0^\beta \int_0^{2\delta} r^{2\gamma-1} \Big[&|\partial_\theta^2 u(r,\theta)|^2 + |\partial_\theta u(r,\theta)|^2 + |r\partial_r \partial_\theta u(r,\theta)|^2 \\ &+ |(r\partial_r)^2 u(r,\theta)|^2 + 2|r\partial_r u(r,\theta)|^2 + |u(r,\theta)|^2 \Big]\, dr\, d\theta,\end{aligned} \tag{4.25}$$

and, since $r < 1$, the right-hand side of (4.24) satisfies

$$C \int_0^\beta \int_0^{2\delta} r^{2\delta-1} |r^2 f(r,\theta)|^2 \, dr\, d\theta \leq \tilde{C} \int_0^\beta \int_0^{2\delta} r^{2\delta-1} |f(r,\theta)|^2 \, dr\, d\theta. \tag{4.26}$$

Finally, expressions (4.25) and (4.26) imply the following estimate:

$$\left\| u; H^2_{\gamma+1}(\mathcal{K}) \right\| \leq C_0 \left\| f; H^0_{\gamma+1}(\mathcal{K}) \right\|. \tag{4.27}$$

At this point, note that if b is not an integer, then the γ in (4.27) can be any integer. Now, if $f \in H^m_{\gamma+1}$, then it follows from (4.18) that

$$\partial_\theta^{2+m} \widehat{u} = -(b+1)^2 \lambda^2 \partial_\theta^m \widehat{u} - (b+1)^2 \partial_\theta^m \widehat{F}.$$

With some slight modification in the computations above, one gets the following proposition:

Proposition 4.2. *If f belongs to $H^m_{\gamma+1}(\mathcal{K})$, where $\gamma - m \neq k/(b+1)$ for $k = 0, \pm 1, \pm 2, \ldots$, then there exists a unique solution $u \in H^{m+2}_{\gamma+1}(\mathcal{K})$ of the problem* (4.16)*. Furthermore, the solution u satisfies the estimate*

$$\left\| u; H^{m+2}_{\gamma+1}(\mathcal{K}) \right\| \leq C_0 \left\| f; H^m_{\gamma+1}(\mathcal{K}) \right\|. \tag{4.28}$$

4.1.3 Asymptotics for functions from $\mathcal{D}(\bar{\Delta})$

Let $u \in C_0^\infty(\mathcal{K} \setminus \{\mathcal{O}\})$ and let $0 \le \chi \le 1$ be a smooth cut-off function such that $\chi \equiv 1$ if $0 < r < \delta < 1/2$, and $\chi \equiv 0$ if $r > 2\delta$, where δ is sufficiently small. Let $\epsilon > 0$ be sufficiently small. Using (4.27), one finds that

$$\left\|\chi u; H_\epsilon^2(\mathcal{K})\right\| \le c_1 \left\|\Delta(\chi u); H_\epsilon^0(\mathcal{K})\right\| \le c_2 \left\|\Delta(\chi u); L^2(\mathcal{K})\right\|,$$

for some constants c_1 and c_2. Noting that $\Delta(\chi u) = (\Delta\chi)u + 2\nabla\chi \cdot \nabla u + \chi(\Delta u)$, one has

$$\left\|\chi u; H_\epsilon^2(\mathcal{K})\right\| \le c_2 \Big(\left\|(\Delta\chi)u; L^2(\mathcal{K})\right\| + 2\left\|\nabla\chi \cdot \nabla u; L^2(\mathcal{K})\right\| + \left\|\chi(\Delta u); L^2(\mathcal{K})\right\| \Big).$$

Definition of χ implies that

$$\int_0^{2\delta} |\chi(\Delta u)|^2 dr \le \left\|\Delta u; L^2(\mathbb{R}^+)\right\|^2.$$

Also, Mean-value theorem for integrals gives

$$\int_\delta^{2\delta} |(\Delta\chi)u|^2 dr \le \delta \max|\Delta\chi|^2 \left\|u; L^2(\mathbb{R}^+)\right\|^2.$$

Finally, using the standard elliptic estimate (see, e.g. [5], Theorem 2.1 in Supplement 2), one finds that

$$\int_\delta^{2\delta} |\nabla\chi \cdot \nabla u|^2 dr \le \tilde{C} \left\|\Delta u; L^2(\mathbb{R}^+)\right\|^2,$$

for some constant $\tilde{C}$. Therefore,

$$\left\|\chi u; H_\epsilon^2(\mathcal{K})\right\| \le C \left(\left\|\Delta u; L^2(\mathcal{K})\right\| + \left\|u; L^2(\mathcal{K})\right\|\right).$$

Since χ was arbitrary, the estimate

$$\left\|u; H_\epsilon^2(\mathcal{K})\right\| \le C \left(\left\|\Delta u; L^2(\mathcal{K})\right\| + \left\|u; L^2(\mathcal{K})\right\|\right) \tag{4.29}$$

holds true.

Now, if u belongs to $\mathcal{D}(\bar{\Delta})$, then there is a sequence $\{u_n\} \subset C_0^\infty(\mathcal{K} \setminus \{\mathcal{O}\})$ such that $u_n \to u$ in the graph norm (2.1). Hence, the a priori estimate (4.29) also holds true if u belongs to $\mathcal{D}(\bar{\Delta})$. Moreover, the estimate (4.29) also implies that if $u \in \mathcal{D}(\bar{\Delta})$ near the tip of $\mathcal{K}$, then u also belongs to $H^2_\epsilon(\mathcal{K})$. Using the standard Sobolev lemma, one has

$$\sup_{1/2 \le |x| \le 1} |u(x)|^2 \le C \sum_{|\alpha| \le 2} \int_{1/2 \le |x| \le 1} r^{2(\epsilon - 2 + |\alpha|)} |\partial_x^\alpha u(x)|^2 dx,$$

with a constant C (not the same C from (4.29)) independent of $u \in H^2_\epsilon(\mathcal{K})$. Thus, for sufficiently small $\delta > 0$,

$$\begin{aligned}
\sum_{|\alpha| \le 2} \int_{\delta/2 \le |x| \le \delta} & r^{2(\epsilon - 2 + |\alpha|)} |\partial_x^\alpha u(x)|^2 dx \\
&= \delta^{2(\epsilon - 1)} \sum_{|\alpha| \le 2} \int_{1/2 \le |x| \le 1} r^{2(\epsilon - 2 + |\alpha|)} |\partial_x^\alpha u(\delta x)|^2 dx \\
&\ge C^{-1} \delta^{2(\epsilon - 1)} \sup_{\delta/2 \le |x| \le \delta} |u(x)|^2,
\end{aligned}$$

and, therefore,

$$u = O(r^{1-\epsilon}) \tag{4.30}$$

for $u \in \mathcal{D}(\bar{\Delta})$ near $\mathcal{O}$. The latter estimate can be improved to $u = O(r)$ in case of conical angles not equal to an integer multiple of 2π.

4.1.4 Elements in $\mathcal{D}(\Delta^*)$

A standard result from Operator theory (see, e.g., [36] Section X.1) states that

$$\mathcal{D}(\Delta^*) = \ker(\Delta^* + i) \oplus \ker(\Delta^* - i) \oplus \mathcal{D}(\bar{\Delta}).$$

Thus, if $u \in \mathcal{D}(\Delta^*)$, one can find $u_{1,2} \in \ker(\Delta^* \pm i)$ and $u_3 \in \mathcal{D}(\bar{\Delta})$ such that $u = u_1 + u_2 + u_3$. So to find the complete asymptotics of $u \in \mathcal{D}(\Delta^*)$, it suffices to find the asymptotics of the functions from $\ker(\Delta^* - i)$ (the asymptotics of the functions from $\ker(\Delta^* + i)$ is done similarly, and the asymptotics of the functions from $\mathcal{D}(\bar{\Delta})$ was obtained in Section 4.1.3).

Let $v \in \ker(\Delta^* - i)$. Then v belongs to both $L^2(\mathcal{K})$ and $C^\infty(\mathcal{K} \setminus \{\mathcal{O}\})$. We prove the following lemma:

Lemma 4.3. *For some $\epsilon > 0$, one has*

$$\int_{\{x \in \mathcal{K}: |x| \leq \epsilon\}} \left(r^4 |\nabla^2 v|^2 + r^2 |\nabla v|^2\right) \, dx < \infty.$$

Proof. It is enough to show that $v \in H_2^2(\mathcal{K})$ near $\mathcal{O}$. Let χ be the cut-off function as in before and consider $v_1 = \chi v$. Then

$$\Delta v_1 = i v_1 + f, \tag{4.31}$$

where $f \in C_0^\infty(\mathcal{K} \setminus \{\mathcal{O}\})$. Let $\kappa, \phi \in C_0^\infty(\mathcal{K} \setminus \{\mathcal{O}\})$ be such that $\phi\kappa = \kappa$, $\operatorname{supp} \kappa \subset \{x : 1/2 < |x| < 2\}$, and $\operatorname{supp} \phi \subset \{x : 1/4 < |x| < 4\}$. Using the standard elliptic estimates, one has

$$\left\| \kappa v_1; H^2(\mathcal{K}) \right\| \leq c \left(\left\| \phi \Delta v_1; L^2(\mathcal{K}) \right\| + \left\| \phi v_1; L^2(\mathcal{K}) \right\| \right). \tag{4.32}$$

Choose a partition of unity $\{\kappa_j\}$ and functions $\phi_j \in C_0^\infty(\mathcal{K} \setminus \{\mathcal{O}\})$ such that $\kappa_j \phi_j = \kappa_j$,

$$\operatorname{supp} \kappa_j \subset \{x : 2^{j-1} < |x| < 2^{j+1}\},$$

$$\operatorname{supp}\phi_j \subset \{x : 2^{j-2} < |x| < 2^{j+2}\},$$

and

$$|D^\alpha \kappa_j| + |D^\alpha \phi_j| \le C_\alpha 2^{-j|\alpha|}.$$

Then for each j, using (4.32) and (4.31), one gets

$$\begin{aligned}
&\sum_{|\alpha|\le 2} \int_0^\infty |(R\partial_R)^{2-|\alpha|}\partial_\theta^{|\alpha|}\kappa_j(R,\theta)v_1(R,\theta)|^2 R\,dR \\
&\quad\le C_1\Big(\int_0^\infty |\phi_j(R,\theta)\Delta v_1(R,\theta)|^2 R\,dR + \int_0^\infty |\phi_j(R,\theta)v_1(R,\theta)|^2 R\,dR\Big) \qquad (4.33)\\
&\quad\le C_2\Big(\int_0^\infty |\phi_j(R,\theta)v_1(R,\theta)|^2 R\,dR + \int_0^\infty |\phi_j(R,\theta)f(R,\theta)|^2 R\,dR\Big).
\end{aligned}$$

Using the change of variable $R = 2^{-j}r$, the left-hand side of (4.33) turns to

$$\begin{aligned}
&\sum_{|\alpha|\le 2} \int_{2^{2j-1}}^{2^{2j+1}} |(2^{-2j}r\partial_r)^{2-|\alpha|}\partial_\theta^{|\alpha|}\kappa_j(r,\theta)v_1(r,\theta)|^2\, 2^{-j}r\, d(2^{-j}r) \\
&\qquad= \sum_{|\alpha|\le 2} \int_{2^{2j-1}}^{2^{2j+1}} 2^{-4j(2-|\alpha|)}|(r\partial_r)^{2-|\alpha|}\partial_\theta^{|\alpha|}\kappa_j(r,\theta)v_1(r,\theta)|^2\, 2^{-2j}r\,dr \qquad (4.34)\\
&\qquad\ge \sum_{|\alpha|\le 2} \int_{2^{2j-1}}^{2^{2j+1}} 2^{-10j-2|\alpha|}r^{2|\alpha|}|(r\partial_r)^{2-|\alpha|}\partial_\theta^{|\alpha|}\kappa_j(r,\theta)v_1(r,\theta)|^2\, r\,dr.
\end{aligned}$$

Since $v_1 \in L^2(\mathcal{K})$ and $f \in C_0^\infty(\mathcal{K}\setminus\{\mathcal{O}\})$, multiplying by $2^{10j+2|\alpha|}$, summing over $j = 0, -1, -2, \dots$ and integrating over $[0,\beta]$ imply that the last expression in (4.33) is finite, while the last expression in (4.34) becomes $\|v_1; H_2^2(\mathcal{K})\|^2$. Since χ is arbitrary, we conclude that $v \in H_2^2(\mathcal{K})$. □

In the preceding proof, one finds that the solution v_1 to the problem

$$(\Delta - i)w = f \quad \text{in } \ \mathcal{K}, \qquad (4.35)$$

with $f \in C_0^\infty(\mathcal{K}\setminus\{\mathcal{O}\})$ belongs to the Sobolev space $H_2^2(\mathcal{K})$. In fact, one may find other solutions to (4.35) in other weighted Sobolev spaces. Moreover, if $v_1 \in H^2_{\gamma_1}(\mathcal{K})$ and $v_2 \in H^2_{\gamma_2}(\mathcal{K})$, where $\gamma_1 < \gamma_2$, are two solutions of (4.35), then one can write

$$\chi v_1 = \chi \sum_k W_k + \chi v_2,$$

for some functions W_k which depends on the eigenvalues λ_k of the pencil $\mathscr{A}(\lambda)$ inside the strip $-\gamma_2 < \mathrm{Re}(\lambda) < -\gamma_1$. To see this, we may assume that $v_1, v_2 \in L^2(\mathcal{K} \setminus B_\epsilon)$, where $B_\epsilon := \{x \in \mathbb{R}^2 : |x| < \epsilon\}$ for sufficiently small $\epsilon > 0$. Then, the function $v_1 - v_2$ solves the homogeneous problem (4.7), where $\rho^2 = i$. Therefore, for some index set $Z \subset \mathbb{Z}$,

$$v_1 - v_2 = \sum_{k \in Z} w_k, \tag{4.36}$$

where $w_k's$ are given by (4.15). Here, $Z := \{k \in \mathbb{Z} : -\gamma_2 < \mathrm{Re}(\lambda_k) < -\gamma_1\}$. Particularly, let $\epsilon > 0$ be such that $n/(b+1) < 1 - \epsilon < (n+1)/(b+1)$ (recall that n is the integer such that $2\pi n < \beta = 2\pi(b+1) \leq 2\pi(n+1)$). Then by (4.36),

$$\begin{aligned}\chi v_1 = \sum_{k=-n}^{-1} d_k K_{-k/(b+1)}(re^{3\pi i/4}) r^{k/(b+1)} \varphi_k(\theta) + c_0 K_0(re^{3\pi i/4}) \\ + \tilde{c}_0 I_0(re^{3\pi i/4}) + \sum_{k=1}^{n} d_k I_{k/(b+1)}(re^{3\pi i/4}) r^{k/(b+1)} \varphi_k(\theta) + \chi v_2,\end{aligned}$$

where v_2 satisfies the asymptotics $v_2 = o(r^{\frac{n}{b+1}})$. Note that in the distinguished local parameter ζ near the vertex $\mathcal{O}$, one has $r^{k/(b+1)}\varphi_k(\theta) = c_k \zeta^k + \tilde{c}_k \bar{\zeta}^{-k}$ for some constants c_k and $\tilde{c}_k$.

4.1.5 Proof of Proposition 2.1: Conclusion

At this point, the function v_1 takes the form (2.7) with remainder R satisfying the asymptotics $R = o(r^{\frac{n}{b+1}}) = o(|\zeta|^n)$. This remainder is smooth away from the vertex and the derivative R' satisfies the asymptotics $R' = o(r^{\frac{n}{b+1}-1})$. It remains to prove that R belongs $\mathcal{D}(\bar{\Delta})$. For this, put $\psi = 1 - \chi$, where χ is as in above. For sufficiently small $\epsilon' > 0$, one has

$$\Delta(\psi(x/\epsilon')R(x)) = \Delta(\psi(x/\epsilon'))R(x) + 2\nabla\psi(x/\epsilon') \cdot \nabla R(x) + \psi(x/\epsilon')\Delta(R(x)).$$

Now, since $\dfrac{n+1}{b+1} > 1$,

$$\left\|(\Delta\psi(x/\epsilon'))R(x); L^2(\mathbb{R}^+)\right\|^2 \leq c\int_{\delta\epsilon'}^{2\delta\epsilon'} |\Delta\psi(x/\epsilon')|^2 r^{\frac{2n+2}{b+1}+1}\,dr$$

$$\leq c'(\epsilon')^{\frac{2n+2}{b+1}+2}\frac{1}{(\epsilon')^4} \leq M_1,$$

for some constant M_1. Similarly,

$$\left\|\nabla\psi(x/\epsilon') \cdot \nabla R(x); L^2(\mathbb{R}^+)\right\|^2 \leq c\int_{\delta\epsilon'}^{2\delta\epsilon'} |\nabla\psi(x/\epsilon')|^2 r^{\frac{2n+2}{b+1}}\,dr$$

$$\leq c'(\epsilon')^{\frac{2n+2}{b+1}+1}\frac{1}{(\epsilon')^2} \leq M_2,$$

for some constant M_2. Moreover,

$$\left\|\psi(x/\epsilon')\Delta R(x); L^2(X)\right\|^2 = \int_X |\psi(x/\epsilon')\Delta R(x)|^2 dx \leq c\left\|\Delta R(x); L^2(X)\right\|^2.$$

Hence, $\Delta(\psi(x/\epsilon')R(x))$ is uniformly bounded in $L^2(X)$ as $\epsilon' \to 0$, for instance, by M. Put $\psi_{\epsilon'} := \psi(\cdot/\epsilon')$. For any test function $w \in \mathcal{D}(\Delta^*)$,

$$\begin{aligned}
|\langle R, \Delta^* w\rangle| &= \lim_{\epsilon'\to 0} |\langle \psi_{\epsilon'} R, \Delta^* w\rangle| \\
&= \lim_{\epsilon'\to 0} |\langle \Delta(\psi_{\epsilon'} R), w\rangle| \\
&\leq \sup_{\epsilon'} \|\Delta(\psi_{\epsilon'} R); L^2(X\backslash\{P\})\|\ \|w; \mathcal{D}(\Delta^*)\| \\
&\leq M\|w; \mathcal{D}(\Delta^*)\|,
\end{aligned}$$

and, therefore, R belongs to $\mathcal{D}((\Delta^*)^*) = \mathcal{D}(\overline{\Delta})$. The proof of Proposition 2.1 is now complete.

4.2 Proof of Proposition 2.2

Let Ω be the symplectic form on the factor space $\mathcal{D}(\Delta^*)/\mathcal{D}(\overline{\Delta})$:

$$\Omega\left([u],[v]\right) := \langle \Delta^* u, \overline{v}\rangle - \langle u, \Delta^* \overline{v}\rangle\,,$$

where $\langle u, v\rangle = \int_X u\overline{v}\, dS$ is the usual hermitian product with volume element

$$dS = \mathbf{m}(\zeta,\bar{\zeta})\frac{d\zeta\wedge d\bar{\zeta}}{-2i} = -\frac{1}{2i}\mathbf{m}(\zeta,\bar{\zeta})|d\zeta|^2.$$

Proposition 2.2. *One has*

$$\Omega\left([u],[v]\right) = \sum_{k=1}^{M} X_k(u)\begin{pmatrix} 0 & -I_{2n_k+1} \\ I_{2n_k+1} & 0 \end{pmatrix} X_k(v)^t \tag{2.9}$$

where $X_k(u) = (\mathfrak{L}_k(u), \mathfrak{H}_{k,1}(u), \dots, \mathfrak{H}_{k,n_k}(u), \mathfrak{A}_{k,1}(u), \dots, \mathfrak{A}_{k,n_k}(u), \mathfrak{c}_k(u), \mathfrak{h}_{k,1}(u), \dots,$ $\mathfrak{h}_{k,n_k}(u), \mathfrak{a}_{k,1}(u), \dots, \mathfrak{a}_{k,n_k}(u))$.

Proof. For $k = 1, \dots, M$, let Γ_k be a sufficiently small disk, oriented clockwise, with radius $\epsilon_k > 0$ and centered at P_k, and such that $\Gamma_k \cap \Gamma_j = \emptyset$ for $k \neq j$. Put $\Gamma = \bigcup_{k=1}^M \Gamma_k$ and $\epsilon_0 = \max \epsilon_k$. Then, for any $u, v \in \mathcal{D}(\Delta^*)$, by using Green's formula, one obtains

$$\begin{aligned}\langle \Delta^* u, \overline{v}\rangle - \langle u, \Delta^*\overline{v}\rangle &= \lim_{\epsilon_0\to 0}\left[\int_{X\setminus\Gamma} v\,\Delta^* u\, dS - \int_{X\setminus\Gamma} u\,\overline{\Delta^*} v\, dS\right] \\ &= \lim_{\epsilon_0\to 0}\Big(2i\int_{\bigcup\Gamma_k} v\,\partial_{\bar{\zeta}} u\, d\bar{\zeta} - 4\int_{X\setminus\Gamma}\partial_{\bar{\zeta}} u\partial_\zeta v\, dS \\ &\quad -2i\int_{\bigcup\Gamma_k} u\,\partial_\zeta v\, d\zeta + 4\int_{X\setminus\Gamma}\partial_\zeta v\partial_{\bar{\zeta}} u\, dS\Big)\end{aligned} \tag{4.37}$$

$$= \lim_{\epsilon_0 \to 0} \sum_{k=1}^{M} 2i \oint_{\Gamma_k} v\, \partial_{\bar\zeta_k} u\, d\bar\zeta - \lim_{\epsilon_0 \to 0} \sum_{k=1}^{M} 2i \oint_{\Gamma_k} u\, \partial_{\zeta_k} v\, d\zeta$$

$$= \lim_{\epsilon_0 \to 0} \sum_{k=1}^{M} 2i \left[\oint_{\Gamma_k} v\, \partial_{\bar\zeta_k} u\, d\bar\zeta - \oint_{\Gamma_k} u\, \partial_{\zeta_k} v\, d\zeta \right].$$

Let $u, v \in \mathcal{D}(\Delta^*)$. Near a conical point P_k, using (2.7), one gets

$$\partial_{\bar\zeta_k} u = \frac{i}{2\sqrt{2\pi}} \mathfrak{L}_k(u) \frac{1}{\bar\zeta_k} - \frac{1}{\sqrt{4\pi}} \sum_{m=1}^{n_k} \mathfrak{A}_{k,m}(u) \frac{\sqrt{m}}{\bar\zeta_k^{m+1}} + \frac{1}{\sqrt{4\pi}} \sum_{m=1}^{n_k} \mathfrak{a}_{k,m}(u) \sqrt{m}\, \bar\zeta_k^{m-1} + \partial_{\bar\zeta_k}(\chi u_0),$$

$$\partial_{\zeta_k} v = \frac{i}{2\sqrt{2\pi}} \mathfrak{L}_k(v) \frac{1}{\zeta_k} - \frac{1}{\sqrt{4\pi}} \sum_{m=1}^{n_k} \mathfrak{H}_{k,m}(v) \frac{\sqrt{m}}{\zeta_k^{m+1}} + \frac{1}{\sqrt{4\pi}} \sum_{m=1}^{n_k} \mathfrak{h}_{k,m}(v) \sqrt{m}\, \zeta_k^{m-1} + \partial_{\zeta_k}(\chi v_0).$$

Hence,

$$u\partial_{\zeta_k} v = \Big(-\frac{1}{4\pi} \mathfrak{L}_k(v) \mathfrak{c}_k(u) - \frac{1}{4\pi} \mathfrak{H}_k(v) \mathfrak{h}_k(u) + \frac{1}{4\pi} \mathfrak{H}_k(u) \mathfrak{h}_k(v) \Big) \frac{1}{\zeta} + \Theta_1(\zeta, \bar\zeta),$$

$$v\partial_{\bar\zeta_k} u = \Big(-\frac{1}{4\pi} \mathfrak{L}_k(u) \mathfrak{c}_k(v) - \frac{1}{4\pi} \mathfrak{A}_k(u) \mathfrak{a}_k(v) + \frac{1}{4\pi} \mathfrak{A}_k(v) \mathfrak{a}_k(u) \Big) \frac{1}{\zeta} + \Theta_2(\zeta, \bar\zeta)$$

where Θ_1 and Θ_2 are functions such that

$$\lim_{\epsilon_0 \to 0} \oint_{\Gamma_k} \Theta_1(\zeta, \bar\zeta)\, d\zeta = \lim_{\epsilon_0 \to 0} \oint_{\Gamma_k} \Theta_2(\zeta, \bar\zeta)\, d\bar\zeta = 0.$$

It follows from Cauchy's integral formula that

$$2i \Big[\oint_{\Gamma_k} v\, \partial_{\bar\zeta_k} u\, d\bar\zeta - \oint_{\Gamma_k} u\, \partial_{\zeta_k} v\, d\zeta \Big]$$

$$= -(2i)(2\pi i) \Big[-\frac{1}{4\pi} \mathfrak{L}_k(u) \mathfrak{c}_k(v) - \frac{1}{4\pi} \mathfrak{A}_k(u) \mathfrak{a}_k(v) + \frac{1}{4\pi} \mathfrak{A}_k(v) \mathfrak{a}_k(u)$$

$$+ \frac{1}{4\pi} \mathfrak{L}_k(v) \mathfrak{c}_k(u) + \frac{1}{4\pi} \mathfrak{H}_k(v) \mathfrak{h}_k(u) - \frac{1}{4\pi} \mathfrak{H}_k(u) \mathfrak{h}_k(v) \Big]$$

$$= -\mathfrak{L}_k(u)\mathfrak{c}_k(v) - \mathfrak{A}_k(u)\mathfrak{a}_k(v) + \mathfrak{A}_k(v)\mathfrak{a}_k(u) + \mathfrak{L}_k(v)\mathfrak{c}_k(u)$$

$$+\mathfrak{H}_k(v)\mathfrak{h}_k(u) - \mathfrak{H}_k(u)\mathfrak{h}_k(v)$$

and, therefore,

$$\langle \Delta^* u, \overline{v}\rangle - \langle u, \Delta^* \overline{v}\rangle = \sum_{k=1}^{M} \Big(\mathfrak{L}_k(v)\mathfrak{c}_k(u) - \mathfrak{L}_k(u)\mathfrak{c}_k(v) + \sum_{m=1}^{n_k} \mathfrak{h}_{k,m}(u)\mathfrak{H}_{k,m}(v)$$

$$- \sum_{m=1}^{n_k} \mathfrak{H}_{k,m}(u)\mathfrak{h}_{k,m}(v) - \sum_{m=1}^{n_k} \mathfrak{A}_{k,m}(u)\mathfrak{a}_{k,m}(v) + \sum_{m=1}^{n_k} \mathfrak{a}_{k,m}(u)\mathfrak{A}_{k,m}(v)\Big).$$

Formula (2.9) follows. □

4.3 Proof of Proposition 2.8

The next proposition immediately follows from (2.17), (2.18), and Roelcke's formula. For completeness, the proof is provided below.

Proposition 2.8. *One has the following explicit expressions for the special growing solutions of the homogeneous Laplace equation* (2.15) *subject to* (2.16)*:*

$$G_{1/\zeta_k^l}(y;0) = -\frac{1}{(l-1)!\,\mathrm{Area}(X)} \int_X \Omega_{y-q}^{(l-1)}(P_k)dS(q) \quad l = 1, \ldots, n_k, \tag{2.21}$$

$$G_{1/\bar{\zeta}_k^l}(y;0) = \overline{G_{1/\zeta_k^l}(y;0)}\,. \tag{2.22}$$

Here the expression $\Omega_{y-q}^{(l-1)}(P_k)$ *should be understood as follows. Write the one form* Ω_{y-q} *in the distinguished local parameter* ζ_k *in a vicinity of the conical point* P_k*:*

$$\Omega_{y-q} = \omega(\zeta_k)d\zeta_k\,.$$

Then

$$\Omega_{y-q}^{(l-1)}(P_k) := \left(\frac{d}{d\zeta_k}\right)^{l-1} \omega(\zeta_k)|_{\zeta_k=0}\,.$$

Moreover, one has the relation

$$\lim_{\lambda\to 0}\left[G_{\log|\zeta_k|}(y;\lambda) - \frac{2\pi}{\mathrm{Area}(X)\lambda}\right] = \frac{1}{\mathrm{Area}(X)^2}\int_X\int_X \mathrm{Re}\int_p^{P_k} \Omega_{y-q}\; dS(p)dS(q)\,. \quad (2.23)$$

Proof. Rewriting (2.6) as

$$2\pi\left(G(x,y) - G(x,q) + G(p,q) - G(p,y)\right) = \mathrm{Re}\int_p^x \Omega_{y-q} \quad (4.38)$$

then integrating the latter with respect to q gives

$$\begin{aligned} 2\pi\,\mathrm{Area}(X)\left(G(x,y) - G(p,y)\right) &= \int_X \mathrm{Re}\int_p^x \Omega_{y-q}\, dS(q) \\ &= \frac{1}{2}\int_X\left(\int_p^x \Omega_{y-q} + \overline{\int_p^x \Omega_{y-q}}\right) dS(q). \end{aligned} \quad (4.39)$$

On the one hand, take $p = P_k$. In the distinguished local parameter ζ_k near P_k, equation (4.39) becomes

$$2\pi\,\mathrm{Area}(X)\left(G(\zeta_k,y) - G(P_k,y)\right) = \frac{1}{2}\int_X\left(\int_0^{\zeta_k}\omega(\tau)\,d\tau + \int_0^{\overline{\zeta_k}}\omega(\tau)\,d\tau\right) dS(q). \quad (4.40)$$

Differentiating the last equation with respect to ζ_k l-times and then evaluating at $\zeta_k = 0$ give

$$\begin{aligned} \left(\frac{\partial}{\partial\zeta_k}\right)^l G(\zeta_k,y) &= \frac{1}{4\pi\,\mathrm{Area}(X)}\int_X\left(\frac{d}{d\zeta_k}\right)^{l-1}\omega(\zeta_k)|_{\zeta_k=0}\, dS(q) \\ &= \frac{1}{4\pi\,\mathrm{Area}(X)}\int_X \Omega_{y-q}^{(l-1)}(P_k)\, dS(q). \end{aligned} \quad (4.41)$$

On the other hand, differentiating (2.17) with respect to ζ_k l-times and then sending ζ_k to 0 give

$$\left(\frac{\partial}{\partial\zeta_k}\right)^l G(\zeta_k,y) = -\frac{(l-1)!}{4\pi}G_{1/\zeta_k^l}(y;0). \quad (4.42)$$

Formula (2.21) follows.

For the second statement, taking $x = P_k$ in Roelcke's formula (2.4) and combining it with (2.18) give (2.23). □

4.4 Proof of Proposition 3.4

The proof of the next proposition is analogous to the discussion in Section 5 of [13]. For self-containment, the details are provided here.

Proposition 3.4. *Introduce the zeta-regularized determinants of the operators $\Delta_F - \lambda$, $\Delta_{sing} - \lambda$, and $\Delta_{hol} - \lambda$ via*

$$\det A = \exp\{-\zeta_A'(0)\},$$

where $\zeta_A(s) = \zeta(s, A)$ is the operator zeta-function of an operator A (without zero modes). Then

$$\det(\Delta_{sing} - \lambda) = \left(\frac{2\pi^2}{27}\right)^2 \det T(\lambda)\, \det(\Delta_F - \lambda) \tag{3.26}$$

for real λ not belonging to the union of the spectra of Δ_F and Δ_{sing}. Similarly

$$\det(\Delta_{hol} - \lambda) = \frac{2\pi^2}{27} \det P(\lambda)\, \det(\Delta_F - \lambda) \tag{3.27}$$

for real λ not belonging to the union of the spectra of the operators Δ_F and Δ_{hol}.

Proof. For $\lambda \in \mathbb{C}$, let $D(\lambda) = \det T(\lambda)$. If λ does not belong to $\operatorname{spec}(\Delta_F) \cup \operatorname{spec}(\Delta_{sing})$, one can write

$$\operatorname{Trace}\left[(\Delta_{sing} - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}\right] = -\frac{D'(\lambda)}{D(\lambda)}. \tag{4.43}$$

Observe that the right-hand side of (4.43) is the logarithmic derivative of D, and extends

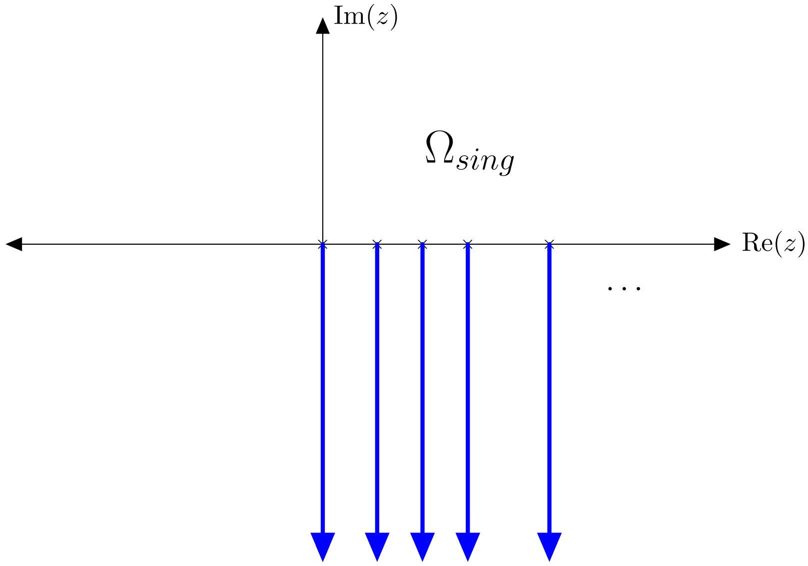

Figure 4.1: $\Omega_{sing} = \mathbb{C} \setminus \{\lambda + it : \ \lambda \in \mathrm{spec}(\Delta_{sing}) \cup \mathrm{spec}(\Delta_F), \ t \in (-\infty, 0]\}$. The $\times$ denotes an eigenvalue of Δ_{sing} or Δ_F.

to a meromorphic function whose poles are the eigenvalues of Δ_{sing} and Δ_F. Each pole has residue equal to the difference $\dim\ker(\Delta_{sing} - \lambda) - \dim\ker(\Delta_F - \lambda)$. Put

$$\Omega_{sing} := \mathbb{C} \setminus \{\lambda + it : \lambda \in \mathrm{spec}(\Delta_F) \cup \mathrm{spec}(\Delta_{sing}),\, t \in (-\infty, 0]\}$$

(see Figure 4.1).

On this domain, define $\tilde{\xi}(\lambda) := -\frac{1}{2\pi i} \log\det T(\lambda)$. By definition,

$$D(\lambda) = \exp(-2\pi i \tilde{\xi}(\lambda)). \tag{4.44}$$

Let $\tilde{\lambda} \in \Omega_{sing}$ with $\mathrm{Re}(\tilde{\lambda}), \mathrm{Im}(\tilde{\lambda}) > 0$. Let C be a sufficiently large negative number; without loss of generality, choose C such that $|\tilde{\lambda}| < -C$. Let $c_{\tilde{\lambda}}$ be the cut consisting of the half-line $(-\infty, C)$ along the real-line and the segment from $\tilde{\lambda}$ to $C + 0i$. Note that for any $\tilde{\lambda}$ and $s \in \mathbb{C}$, the functions $\lambda \mapsto (\lambda - \tilde{\lambda})^{-s}$ is well-defined whenever $\lambda - \tilde{\lambda}$ is a positive real number. This function can be extended to a holomorphic function on the complements of $c_{\tilde{\lambda}}$. Furthermore, as λ tends to the cut $c_{\tilde{\lambda}}$ from above or from below, the following equality holds:

$$\lim_{\lambda \downarrow c_{\tilde{\lambda}}} e^{-i\pi s}(\lambda - \tilde{\lambda})^{-s} = \lim_{\lambda \uparrow c_{\tilde{\lambda}}} e^{i\pi s}(\lambda - \tilde{\lambda})^{-s}.$$

Denote the equal limits by $(\lambda - \tilde{\lambda})_0^{-s}$.

Let $\epsilon > 0$ be sufficiently small. Choose $0 < A \notin \operatorname{spec}(\Delta_F) \cup \operatorname{spec}(\Delta_{sing})$ sufficiently large. Let γ be the contour consisting of circles with centers $\lambda \in \operatorname{spec}(\Delta_F) \cup \operatorname{spec}(\Delta_{sing})$, each has radius ϵ, and

$$A_\epsilon := \{x \pm \epsilon i : x \geq A\} \cup \left\{A + \epsilon e^{i\theta} : \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}.$$

Let $c_{\tilde{\lambda},\epsilon}$ be the contour

$$c_{\tilde{\lambda},\epsilon} := \{z \pm \epsilon i : z \in c_{\tilde{\lambda}}\} \cup \left\{\tilde{\lambda} + \epsilon e^{i\theta} : \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}.$$

One can choose the contours so that they do not intersect. Denote by $\tilde{c}_1$ the part of $c_{\tilde{\lambda},\epsilon}$ with real part less than C and denote by $\tilde{c}_2$ the part of $c_{\tilde{\lambda},\epsilon}$ with real part greater than or equal to C. (See Figure 4.2.)

If $\operatorname{Re}(s) > 1$, using a simple change of integration, one gets

$$\zeta\left(s, \Delta_{sing} - \tilde{\lambda}\right) = \frac{1}{2\pi i} \operatorname{Trace}\left(\int_\gamma \left(\lambda - \tilde{\lambda}\right)^{-s} (\Delta_{sing} - \lambda)^{-1} \, d\lambda\right).$$

Furthermore, noting that the contribution of a large circle centered at $\tilde{\lambda}$ vanishes as the radius of the circle increases, Cauchy integral formula implies that

$$\zeta\left(s, \Delta_{sing} - \tilde{\lambda}\right) = \frac{1}{2\pi i} \operatorname{Trace}\left(\int_{c_{\tilde{\lambda},\epsilon}} \left(\lambda - \tilde{\lambda}\right)^{-s} (\Delta_{sing} - \lambda)^{-1} \, d\lambda\right).$$

One also gets similar formulas for Δ_F (and Δ_{hol}).

Since the difference $(\Delta_{sing} - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}$ is of trace class for $\operatorname{Re}(s) > 1$, the

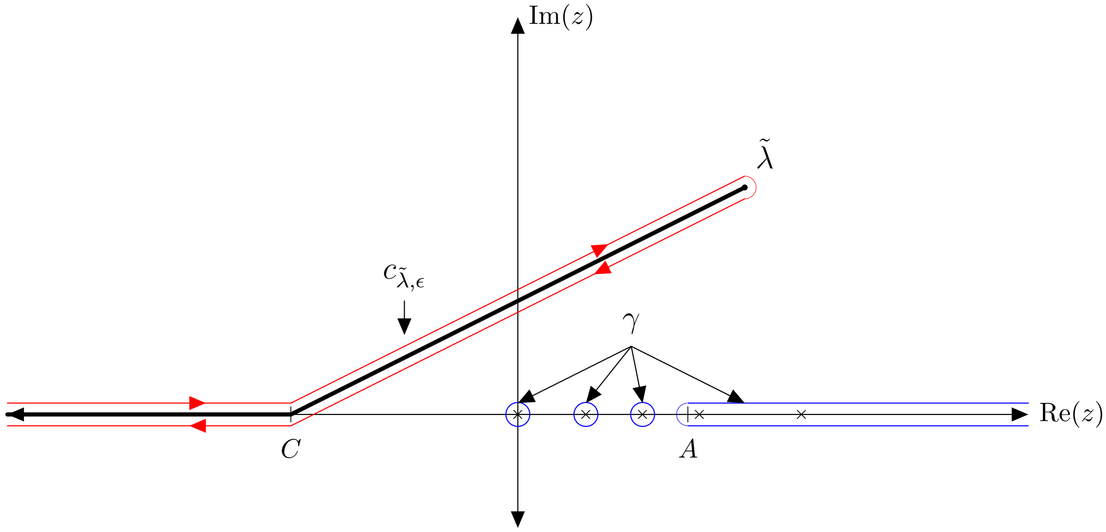

Figure 4.2: Contour for the integration

contour integration and the trace operator can be interchanged so that one obtains

$$\zeta(s, \Delta_{sing} - \tilde{\lambda}) - \zeta(s, \Delta_F - \tilde{\lambda}) = \frac{1}{2\pi i} \int_{c_{\tilde{\lambda},\epsilon}} (\lambda - \tilde{\lambda})^{-s} \,\mathrm{Trace}\left((\Delta_{sing} - \lambda)^{-1} - (\Delta_F - \lambda)^{-1}\right) d\lambda.$$

Using the definition of $\tilde{\xi}$ and the decomposition of $c_{\tilde{\lambda},\epsilon}$ into $\tilde{c}_1$ and $\tilde{c}_2$, the last equation becomes

$$\begin{aligned} \zeta(s, \Delta_{sing} - \tilde{\lambda}) - \zeta(s, \Delta_F - \tilde{\lambda}) &= \int_{c_{\tilde{\lambda},\epsilon}} (\lambda - \tilde{\lambda})^{-s} \tilde{\xi}'(\lambda)\, d\lambda \\ &= \left(\int_{\tilde{c}_1} + \int_{\tilde{c}_2} \right) (\lambda - \tilde{\lambda})^{-s} \tilde{\xi}'(\lambda)\, d\lambda. \end{aligned} \tag{4.45}$$

Denote by $\tilde{\zeta}_k$ the contour integration along $\tilde{c}_k$ in (4.45). Observe that $\tilde{\zeta}_2$ extends to an entire function of s. Moreover, for any s with $\mathrm{Re}(s) < 1$, passing to the limit as ϵ goes to zero gives

$$\zeta_2(s) := \lim_{\epsilon \to 0} \tilde{\zeta}_2(s) = -2i \sin(\pi s) \int_C^{\tilde{\lambda}} (\lambda - \tilde{\lambda})_0^{-s} \tilde{\xi}'(\lambda)\, d\lambda. \tag{4.46}$$

Lemma 4.4. *Let* $\rho : \mathbb{C} \times \{|z| < 1\}$ *be defined by* $\rho(s, z) = (1 - z)^{-s} - 1$ *and let* $0 < r < 1$

and $R > 0$. For any $|z| \le r$ and $|s| \le R$, the following estimate holds:

$$|\rho(s,z)| \le \frac{\exp\left(\frac{rR}{1-r}\right)}{1-r}|s|\,|z|.$$

Proof. Consider the absolutely convergent series representation of ρ: for $|z| < 1$,

$$\rho(s,z) = \sum_{k=1}^{\infty} \frac{(-s)^k\,[\log(1-z)]^k}{k!}.$$

Observe that for $|z| \le r < 1$,

$$|\log(1-z)| \le \sum_{k=1}^{\infty} \frac{|z|^k}{k} \le |z| \sum_{k=0}^{\infty} r^k = \frac{|z|}{1-r}.$$

Therefore, for $|z| \le r < 1$ and $|s| \le R$,

$$|\rho(s,z)| \le \sum_{k=1}^{\infty} \frac{|s|^k |\log(1-z)|}{k!} \le \sum_{k=1}^{\infty} \frac{|s|^k \left(\frac{|z|}{1-r}\right)^k}{k!} = \exp\left(\frac{|s||z|}{1-r}\right) - 1.$$

Finally, observe that

$$\exp\left(\frac{|s||z|}{1-r}\right) - 1 = \int_0^{\frac{|s||z|}{1-r}} e^{\tau}\,d\tau \le \exp\left(\frac{|s||z|}{1-r}\right)\frac{|s||z|}{1-r} \le \exp\left(\frac{Rr}{1-r}\right)\frac{|s||z|}{1-r},$$

and the conclusion of the lemma follows. □

If $\lambda \in \mathbb{C} \setminus (-\infty, C)$ with $\mathrm{Re}(\lambda) < C$, then

$$\left|\tilde{\lambda}/\lambda\right| < \frac{-C}{\sqrt{\mathrm{Re}(\lambda)^2 + \mathrm{Re}(\lambda)^2}} < 1,$$

and so there exists $0 < r < 1$ such that $|\tilde{\lambda}/\lambda| \le r$. Using the previous lemma, one can write

$$\left(\lambda - \tilde{\lambda}\right)^{-s} = \lambda^{-s}\left(1 + \rho\left(s, \tilde{\lambda}/\lambda\right)\right). \tag{4.47}$$

The function $\tilde{\zeta}_1$ can be written as

$$\tilde{\zeta}_1(s) = \left(\int_{-\infty+\epsilon i}^{C+\epsilon i} - \int_{-\infty-\epsilon i}^{C-\epsilon i}\right)(\lambda - \tilde{\lambda})^{-s}\tilde{\xi}'(\lambda)\,d\lambda.$$

Using (4.47) and sending ϵ to zero yield

$$\begin{aligned}\int_{-\infty+\epsilon i}^{C+\epsilon i}(\lambda-\tilde{\lambda})^{-s}\tilde{\xi}'(\lambda)\,d\lambda &= \int_{-\infty+\epsilon i}^{C+\epsilon i}\lambda^{-s}\left(1+\rho\left(s,\tilde{\lambda}/\lambda\right)\right)\tilde{\xi}'(\lambda)\,d\lambda \\ &= \int_{-\infty}^{C}(\lambda+\epsilon i)^{-s}\left(1+\rho\left(s,\frac{\tilde{\lambda}}{\lambda+\epsilon i}\right)\right)\tilde{\xi}'(\lambda+\epsilon i)\,d\lambda \\ &\overset{\epsilon\to 0}{=} e^{-i\pi s}\int_{-\infty}^{C}|\lambda|^{-s}\left(1+\rho\left(s,\tilde{\lambda}/\lambda\right)\right)\tilde{\xi}'(\lambda)\,d\lambda.\end{aligned}$$

Note that this is possible since both ρ and $\tilde{\xi}'$ are continuous functions of λ. Similarly,

$$\int_{-\infty-\epsilon i}^{C-\epsilon i}(\lambda-\tilde{\lambda})^{-s}\tilde{\xi}'(\lambda)\,d\lambda \overset{\epsilon\to 0}{=} e^{i\pi s}\int_{-\infty}^{C}|\lambda|^{-s}\left(1+\rho\left(s,\tilde{\lambda}/\lambda\right)\right)\tilde{\xi}'(\lambda)\,d\lambda.$$

Thus, passing to the limit as $\epsilon \to 0$,

$$\zeta_1(s) := -2i\sin(\pi s)\left(\int_{-\infty}^{C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\,d\lambda + \int_{-\infty}^{C}|\lambda|^{-s}\rho\left(s,\tilde{\lambda}/\lambda\right)\tilde{\xi}'(\lambda)\,d\lambda\right). \qquad (4.48)$$

Therefore, as ϵ tends to zero, (4.45) becomes

$$\zeta(s,\Delta_{sing}-\tilde{\lambda})-\zeta(s,\Delta_F-\tilde{\lambda}) = -2i\sin(\pi s)\int_{-\infty}^{C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\,d\lambda + R_C(s,\tilde{\lambda})+\zeta_2(s)$$

where

$$R_C(s,\tilde{\lambda}) = -2i\sin(\pi s)\int_{-\infty}^{C}|\lambda|^{-s}\rho\left(s,\tilde{\lambda}/\lambda\right)\tilde{\xi}'(\lambda)\,d\lambda. \qquad (4.49)$$

For the moment, consider the function $R_C(s,\tilde{\lambda})$. From (4.49), $R_C(0,\tilde{\lambda}) = 0$. Meanwhile, for $\mathrm{Re}(\lambda) < C$, there exists some constant K that depends only on C and $\tilde{\lambda}$ such that

$$\left||\lambda|^{-s}\rho\left(s,\tilde{\lambda}/\lambda\right)\tilde{\xi}'(\lambda)\right| \le K|s||\lambda|^{-\mathrm{Re}(s)-2}$$

and the preceding estimate is uniform for $|s| \le R$. Indeed, the identity (3.23) implies that

$$-2\pi i\tilde{\xi}'(\lambda) = \frac{2}{|\lambda|} + O(|\lambda|^{-1}), \tag{4.50}$$

and using Lemma 4.4, one obtains the estimate. This implies that R_C can be analytically extended to $\mathrm{Re}(s) > -1$. Moreover, for $\mathrm{Re}(s) > -1$,

$$\begin{aligned}\frac{\partial}{\partial s}R_C(s,\tilde{\lambda}) \;&= 2\pi i\cos(\pi s)\int_{-\infty}^{C}|\lambda|^{-s}\rho\left(s,\tilde{\lambda}/\lambda\right)\tilde{\xi}'(\lambda)\,d\lambda\\ &+2i\sin(\pi s)\int_{-\infty}^{C}|\lambda|^{-s}(-\log\lambda)\rho\left(s,\tilde{\lambda}/\lambda\right)\tilde{\xi}'(\lambda)\,d\lambda\\ &+2i\sin(\pi s)\int_{-\infty}^{C}|\lambda|^{-s}\rho\left(s,\tilde{\lambda}/\lambda\right)(-\log(1-\tilde{\lambda}/\lambda))\tilde{\xi}'(\lambda)\,d\lambda\,,\end{aligned}$$

and since $\rho(0,\tilde{\lambda}/\lambda) = 0$, it follows that $\frac{\partial}{\partial s}R_C(s,\tilde{\lambda})\Big|_{s=0} = 0$. Observe that R_C can also be written as

$$\begin{aligned}R_C(s,\tilde{\lambda}) &= \zeta(s,\Delta_{sing}-\tilde{\lambda}) - \zeta(s,\Delta_F-\tilde{\lambda}) - \zeta_2(s)\\ &\quad + 2i\sin(\pi s)\int_{-\infty}^{C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\,d\lambda.\end{aligned} \tag{4.51}$$

Now, observe that the last term of (4.51) can be written as

$$\begin{aligned}&2i\sin(\pi s)\int_{-\infty}^{C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\,d\lambda\\ &\quad= \frac{\sin(\pi s)}{\pi}\Big[\int_{-\infty}^{C}|\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda)+\frac{2}{|\lambda|}\right)d\lambda - \int_{-\infty}^{C}2|\lambda|^{-s-1}d\lambda\Big]\\ &\quad= \frac{\sin(\pi s)}{\pi}\int_{-\infty}^{C}|\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda)+\frac{2}{|\lambda|}\right)d\lambda + \frac{2(-C)^{-s}\sin(\pi s)}{\pi s}.\end{aligned}$$

Thanks to (4.50), one can conclude that all the terms in (4.51) are regular at $s = 0$.

Employing the last observation, equation (4.51) can be written as

$$\begin{aligned}\zeta(s,\Delta_{sing}-\tilde{\lambda}) \; &= \zeta(s,\Delta_F-\tilde{\lambda}) - \frac{\sin(\pi s)}{\pi}\int_{-\infty}^{C}|\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda)+\frac{2}{|\lambda|}\right)d\lambda \\ &\quad -\frac{2(-C)^{-s}\sin(\pi s)}{\pi s}+\zeta_2(s)+R_C(s,\tilde{\lambda}).\end{aligned} \tag{4.52}$$

Denote by $h_C(s)$ the integral $\int_{-\infty}^{C}|\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda)+\frac{2}{|\lambda|}\right)d\lambda$. Differentiating (4.52) with respect to s and then evaluating at $s=0$, one gets

$$\zeta'(0,\Delta_{sing}-\tilde{\lambda}) = \zeta'(0,\Delta_F-\tilde{\lambda}) + h_C(0) + \zeta_2'(0) + 2\log(-C).$$

Meanwhile,

$$\zeta_2'(0) = -2\pi i\left(\tilde{\xi}(\tilde{\lambda})-\tilde{\xi}(C)\right) = -2\pi i\tilde{\xi}(\tilde{\lambda}) - \log D(C)$$

and

$$h_C(0) = \int_{-\infty}^{C}\left(2\pi i\tilde{\xi}'(\lambda)+\frac{2}{|\lambda|}\right)d\lambda.$$

Furthermore, it follows from (4.50) that $h_C(0)$ vanishes for sufficiently large $-C$. Thus, sending C to $-\infty$ and recalling (4.44) yield

$$\zeta'(0,\Delta_{sing}-\tilde{\lambda}) - \zeta'(0,\Delta_F-\tilde{\lambda}) = \log\left(\frac{2\pi^2}{27}\right)^2 - 2\pi i\tilde{\xi}(\tilde{\lambda}) = \log\left(\frac{2\pi^2}{27}\right)^2 + \log\det T(\tilde{\lambda}),$$

and, therefore,

$$\frac{\det_\zeta\left(\Delta_{sing}-\tilde{\lambda}\right)}{\det_\zeta\left(\Delta_F-\tilde{\lambda}\right)} = \left(\frac{2\pi^2}{27}\right)^2 \det T(\tilde{\lambda}).$$

Analogously, the second statement can be obtained by doing similar calculations as in above. The first change in the computations starts at equation (4.50): one gets the asymptotics

$$-2\pi i\tilde{\xi}'(\lambda) = \frac{1}{|\lambda|} + O(|\lambda|^{-1}).$$

By making all the necessary modifications, one obtains the relations

$$\frac{\det_\zeta\left(\Delta_{hol}-\tilde{\lambda}\right)}{\det_\zeta\left(\Delta_F-\tilde{\lambda}\right)}=\frac{2\pi^2}{27}\,\det P(\tilde{\lambda}).$$

□

4.5 Proof of identities (3.11)

Write the special growing solutions $G_\star(\cdot;\lambda)$ as

$$\begin{aligned} G_\star(\cdot;\lambda) &= \star + S^{\star,1}(\lambda) + \sum_{m=1}^{2}\left(S^{\star,\zeta^m}(\lambda)\zeta^m + S^{\star,\bar{\zeta}^m}(\lambda)\bar{\zeta}^m\right) + o(|\zeta|^2) \\ &= F_\star + u_\star(\cdot;\lambda), \end{aligned}$$

where $F_\star$ is the principal part of $G_\star(\cdot;\lambda)$. Note that $u_\star(\cdot;\lambda)$ belongs to the domain of the Friedrichs extension Δ_F. Recalling the construction of the special growing solutions (see proof of Proposition 2.5), one has the relation

$$(\Delta_F-\lambda)u_\star(\cdot;\lambda) = -(\Delta^*-\lambda)F_\star.$$

Differentiating the last equation with respect to λ, one gets

$$(\Delta_F-\lambda)\partial_\lambda u_\star(\cdot;\lambda) = F_\star + u_\star(\cdot;\lambda) = G_\star(\cdot;\lambda).$$

Thus, using the identity (3.7),

$$\begin{aligned} \left\langle G_\star(\cdot;\lambda), G_{1/\bar{\zeta}}(\cdot;\bar{\lambda})\right\rangle &= \left\langle(\Delta_F-\lambda)\partial_\lambda u_\star(\cdot;\lambda), G_{1/\bar{\zeta}}(\cdot;\bar{\lambda})\right\rangle \\ &= \sqrt{4\pi}\mathfrak{h}_1(\partial_\lambda u_\star(\cdot;\lambda)) \\ &= \sqrt{4\pi}\,\sqrt{4\pi}\frac{d}{d\lambda}S^{\star,\zeta}(\lambda), \end{aligned}$$

and the first identity of (3.11) follows. The rest are done similarly.

4.6 Proof of the elementary relation (3.13)

Denote by $\Lambda_{\mu\nu}$ the rank-one operator on L^2:

$$\Lambda_{\mu\nu} f = G_\mu(\cdot;\lambda)\left\langle f, G_\nu(\cdot;\lambda)\right\rangle.$$

Let $\{e_n\}$ be an orthonormal basis in L^2. Then

$$\begin{aligned}
\operatorname{Trace}(\Lambda_{\mu\nu}) &= \sum_n \langle \Lambda_{\mu\nu}\, e_n, e_n\rangle \\
&= \sum_n \Big\langle G_\mu(\cdot;\lambda)\left\langle e_n, G_\nu(\cdot;\lambda)\right\rangle, e_n \Big\rangle \\
&= \sum_n \langle G_\mu(\cdot;\lambda), e_n\rangle \left\langle e_n, G_\nu(\cdot;\lambda)\right\rangle \\
&= \sum_n \Big\langle \langle G_\mu(\cdot;\lambda), e_n\rangle\, e_n, G_\nu(\cdot;\lambda) \Big\rangle \\
&= \Big\langle \sum_n \langle G_\mu(\cdot;\lambda), e_n\rangle\, e_n, G_\nu(\cdot;\lambda) \Big\rangle \\
&= \left\langle G_\mu(\cdot;\lambda), G_\nu(\cdot;\lambda)\right\rangle.
\end{aligned}$$

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Appendix A

Krein's formula

In this section, we provide a little discussion on the derivation of Krein's formula. See [4], Appendix A, for the details.

Let $\mathcal{H}$ be a Hilbert space and let A be a densely defined, closed, and symmetric operator on $\mathcal{H}$ with deficiency indices $n_+ = n_- = n$, where $n \in \mathbb{N}$. Let A_1 and A_2 be self-adjoint extensions of A. Let $\lambda \in (\mathrm{spec}(A_1) \cup \mathrm{spec}(A_2))^c$. One can decompose $\mathcal{H}$ into

$$\mathcal{H} = \overline{\mathrm{ran}(A - \lambda)} \oplus \ker(A^* - \overline{\lambda}) = \overline{\mathrm{ran}(A - \overline{\lambda})} \oplus \ker(A^* - \lambda).$$

Denote by $R_\lambda(A_m)$ the resolvent operator $(A_m - \lambda)^{-1}$ of A_m at λ.

Proposition A.1. *The operator $R_\lambda(A_1) - R_\lambda(A_2)$ is finite-rank. Furthermore, the operator sends ran$(A - \lambda)$ to $\{0\}$, and $\ker(A^* - \overline{\lambda})$ to $\ker(A^* - \lambda)$.*

Proof. By assumption, $\dim\ker(A^* - \lambda) = n$. Thus, it is enough to show that the second statement is true. On the one hand, let $f \in \mathrm{ran}(A - \lambda)$ and let $x \in \mathcal{D}(A)$ such that

$f = (A - \lambda)x$. Then

$$\left[(A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}\right] f = \left[(A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}\right](A - \lambda)x = x - x = 0,$$

noting that A_1 and A_2 are extensions of A. On the other hand, let $f \in \ker(A^* - \overline{\lambda})$. For any $h \in \operatorname{ran}(A - \overline{\lambda})$,

$$\left\langle \left[(A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}\right] f, h\right\rangle = \left\langle f, \left[(A_1 - \overline{\lambda})^{-1} - (A_2 - \overline{\lambda})^{-1}\right] h\right\rangle = \langle f, 0\rangle = 0.$$

Thus, $[(A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}] f \in \operatorname{ran}(A - \overline{\lambda})^{\perp} = \ker(A^* - \lambda)$. This completes the proof. □

Now, fix $\lambda \in (\operatorname{spec}(A_1) \cup \operatorname{spec}(A_2))^c$. Choose a basis for $\ker(A^* - \lambda)$: $\{g_1(\lambda), ..., g_n(\lambda)\}$, and a basis for $\ker(A^* - \overline{\lambda})$: $\{g_1(\overline{\lambda}), ..., g_n(\overline{\lambda})\}$. For any $h \in \mathcal{H}$,

$$[R_\lambda(A_1) - R_\lambda(A_2)] f = \sum_{k=1}^{n} c_k(f) g_k(\lambda)$$

where $c_k(\cdot)$ is a bounded linear functional on $\mathcal{H}$. Thus, Riesz' representation theorem implies that $c_k(f) = \langle f, h_k\rangle$ for some $h_k \in \mathcal{H}$. From the previous proposition, if $f \in \operatorname{ran}(A - \lambda)$, then $\langle h_k, f\rangle = 0$. Hence $h_k \in \ker(A^* - \overline{\lambda})$ and therefore, it can be written as

$$h_k = \sum_{s=1}^{n} \overline{p}_{k,s}(\lambda) g_s(\overline{\lambda})$$

for some constants $\overline{p}_{k,s}(\lambda)$. Therefore,

$$[R_\lambda(A_1) - R_\lambda(A_2)]\, f = \sum_{k,s=1}^{n} p_{k,s}(\lambda) \left\langle f, g_s(\overline{\lambda}) \right\rangle g_k(\lambda).$$

We just proved Krein's formula:

Theorem A.2 (Krein's formula for deficiency indices $n > 1$)**.** *Let A, A_1, and A_2 be as in above. If $\lambda \in (\operatorname{spec}(A_1) \cup \operatorname{spec}(A_2))^c$, then*

$$R_\lambda(A_1) - R_\lambda(A_2) = \sum_{k,s=1}^{n} p_{k,s}(\lambda) \left\langle \cdot, g_s(\overline{\lambda}) \right\rangle g_k(\lambda) \tag{A.1}$$

Definition A.3. *The self-adjoint extensions A_1 and A_2 of A are said to be* **relatively prime** *if $\mathcal{D}(A_1) \cap \mathcal{D}(A_2) = \mathcal{D}(A)$.*

Proposition A.4. *Let $\mathbf{P}(\lambda) = \|p_{k,s}(\lambda)\|$. If A_1 and A_2 are relatively prime, then $\det \mathbf{P}(\lambda) \neq 0$.*

Proof. For each $k = 1, 2, ..., n$, set

$$h_k = \sum_{s=1}^{n} \overline{p}_{k,s}(\lambda) g_s(\overline{\lambda}).$$

If $\det \mathbb{P}(\lambda) = 0$, then $h_1, ..., h_n$ are linearly dependent. There exists $h \in \ker(A^* - \overline{\lambda})$, $h \neq 0$ such that $h \perp h_k$ for all $k = 1, ..., n$. Hence,

$$[R(\lambda, A_1) - R(\lambda, A_2)]\, h = \sum_{k=1}^{n} \langle h, h_k \rangle\, g_k(\lambda) = 0$$

and so

$$R(\lambda, A_1)h = R(\lambda, A_2)h.$$

The left-hand side of the last equation is in $\mathcal{D}(A_1)$ while the right-hand side is in $\mathcal{D}(A_2)$. Since A_1 and A_2 are relatively prime, $R(\lambda, A_1) \in \mathcal{D}(A)$. Thus, $(A - \lambda)(A_1 - \lambda)^{-1}h = h$ so that $h \in \operatorname{ran}(A - \lambda)$. But $h \in \ker(A^* - \overline{\lambda}) = [\operatorname{ran}(A - \lambda)]^{\perp}$ implying that $h = 0$, a contradiction. □

Remark A.5. *In equation* (A.1)*, g_k and $p_{k,s}$ can be chosen as regular functions in* $\mathbb{C}\setminus\mathbb{R}$*. Furthermore, if $\lambda_0 \in [\operatorname{spec}(A)]^c$, then*

$$g_k(\lambda) = g_k(\lambda_0) + (\lambda - \lambda_0)R_\lambda(A_1)g_k(\lambda_0)$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$.